Игра в кварки

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http://skopenkov.ru/courses/quarks-17.html

В этом курсе в элементарной игровой форме мы познакомимся с важными идеями теории поля, описывающей взаимодействия элементарных частиц. Это позволит понять не только физику, но и такие разделы математики, как дифференциальная геометрия и комплексный анализ. Для каждой изучаемой теории, каждого нового понятия мы постараемся показать, как они естественно возникают при решении практических задач, к каким задачам применяются дальше. Благодаря этому большинство объектов становятся наглядными и простыми.

Материал будет изучаться в виде решения задач участниками, с подробными указаниями и последующим разбором на занятии. Никаких предварительных знаний физики не требуется. Первые занятия доступны школьникам.

Примерная программа.

1. Игрушечная модель калибровочной теории на решетке: обмен товарами между городами. Связь с магнитным полем. Квантование: случайные курсы обмена товарами. Точное решение 1- и 2-мерной калибровочной теории на решетке. Численные эксперименты в размерности 3 и 4. Пример неабелевой калибровочной теории. Пленение кварков. Суть проблемы о решении уравнения Янга-Миллса (одной из "проблем тысячелетия").

2. Математическая модель электрической цепи - простейшая модель теории поля на решетке. Существование и единственность потенциала в электрической цепи. Принцип максимума. Сохранение энергии. Вариационный принцип. Магнитное поле. Связь с игрушечной калибровочной теорией. Дискретные гармонические и дискретные аналитические функции. Электромагнитное поле*. Дискретные уравнения Максвелла*.

3. Шашки Фейнмана - простейшая модель электрона. Спин. Дискретное уравнение Дирака.* Сходимость шашек Фейнмана к теории Дирака*.

[1] M. Creutz, Quarks, Gluons and Lattices, Cambridge Univ. Press, 1983 - Science - 169 pp.

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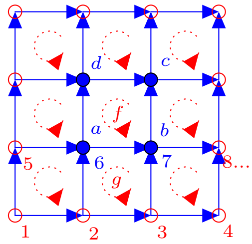

A historical remark. In 1970s K. Wilson introduced *lattice gauge theory* as a computational tool for gauge theory describing all known interactions between elementary particles except gravity; see [1]. Further the model allowed to determine the proton mass with an error less than 2% in a sense. Using the model, he proved *confinement of quarks* in the limit of large interaction constant. The case of arbitrary interaction constant remains a famous open problem[1].

Toy model. Several cities are connected by roads in the shape of an $M \times N$ grid; see the figure. Each city has its own type of goods. E.g., city a has apples and city b has bananas. For two neighboring cities a and b an exchange rate $U(ab)$ is fixed, e.g., 2 banana for an apple. The rate is symmetric, i.e., $U(ba) = U(ab)^{-1}$: one gets back an apple for 2 banana.

A cunning citizen can travel and exchange along a square $abcd$ to multiply his initial amount of goods by a factor of $U(ab)U(bc)U(cd)U(da)$. The total speculation profit is measured by the quantity

$$\mathcal{S}[U] := \sum_{\text{faces } abcd} L(U(ab)U(bc)U(cd)U(da)).$$

where $L(x)$ is a function vanishing at $x = 1$ and positive for $x \neq 1$.

The king can set exchange rates except those on the boundary of the grid. He sets them to minimize the quantity $\mathcal{S}[U]$. The resulting rates are called *optimal*.

Denote by W the factor multiplying the initial amount of goods for a counterclockwise travel around the whole boundary.

Particular case (A): $L(x) = \log_2^2 x$ and the fixed rates at the boundary are

$$U(ab) = \begin{cases} 2, & \text{if } ab \text{ is on the northern border of the grid;} \\ 1/2, & \text{if } ab \text{ is on the southern border of the grid;} \\ 1, & \text{if } ab \text{ is on the eastern or western border of the grid.} \end{cases}$$

The change of variables $A(ab) := \log_2 U(ab)$ simplifies the speculation profit function a lot:

$$\mathcal{S}[A] := \sum_{\text{faces } abcd} (A(ab) + A(bc) + A(cd) + A(da))^2.$$

The new variables satisfy $A(ab) = -A(ba)$. Denote $F(abcd) := A(ab) + A(bc) + A(cd) + A(da)$.

1. In case (A) find the optimal rates and $A(ab)$, $F(abcd)$, W, $S[U]$ for the grid:

$$1 \times 1; \quad 1 \times 2; \quad 1 \times 3; \quad 1 \times N; \quad 2 \times 2.$$

A physical interpretation. Roughly, for $L(x) = \log_2^2 x$ the values $A(ab)$ at the boundary represent *electric current* (with the sign meaning the direction), $F(abcd)$ represent *magnetic flux* generated by the current, $S[U]$ represents the energy of the magnetic field. Each system tries to minimize its total energy (moving the conductors with currents, if their positions are not fixed).

2. a) Do parallel conductors with opposite currents magnetically attract or repulse?
b)* And if the current directions are the same?
c) Is the amount of magnetic energy freed by moving two parallel conductors with opposite currents far away from each other finite or infinite?

3. a) For which values of W the king can achieve $S[U] = 0$?
b) Assume that $S[U] = 0$. Can the citizen get a profit by moving along a closed path?
c) For which values of M and N the optimal rates are unique?

4. a) In case (A), how M, N, W and the minimal speculation profit are related?
b) In case (A), when the grid $M \times N$ has smaller speculation profit than the grid $K \times L$?
c) Does a loop with current try to increase or decrease its area in a magnetic field?

[1]Actually one of the Millenium problems, the essence of which we also are going to explain in the project.

5. (Gauss–Bonnet) Consider a) a cube; b) a regular tetrahedron; c) an octahedron. Two vectors lying in neighboring faces are *parallel*, if they form equal "oriented angles" with the common side of the faces. Let f_1, $f_2, \dots, f_k, f_1$ be all the faces around a vertex v in the natural order. Start with a vector $\vec{e}_1 \subset f_1$ and take the vector $\vec{e}_2 \subset f_2$ parallel to $\vec{e}_1$, the vector $\vec{e}_3 \subset f_3$ parallel to $\vec{e}_2, \dots$, the vector $\vec{e}_{k+1} \subset f_1$ parallel to $\vec{e}_k$. Let ϕ_v be the oriented angle between $\vec{e}_{k+1}$ and $\vec{e}_1$. Find the sum of ϕ_v over all vertices v.

A brief introduction to probability here.

Quantization. Now let the collection of rates $U(ab) \in \{+1, -1\}$ be random with the probability of a collection U proportional to $2^{-S[U]}$, where $L(x) = \begin{cases} 0, & \text{if } x = 1; \\ 1, & \text{if } x = -1. \end{cases}$

Physical interpretation. Roughly, the energy of the electromagnetic field between two quarks at distance N is $E_M(N) = -\frac{1}{M}\log_2 \mathrm{E}(W)$, where $\mathrm{E}(W)$ is the expectation of W.

6. Compute the expectation $\mathrm{E}(W)$ of W and the energy $E_1(N)$ for the grid:

$$1 \times 1; \quad 1 \times 2; \quad 1 \times 3; \quad 1 \times N.$$

7. (Quarks confinement in 1-dimensional space.) Is the amount of energy $E_1(N)$ required to move two quarks far away from each other finite or infinite?

8. (Wilson's area law) Compute $\mathrm{E}(W)$ and the energy $E_M(N)$ for the grid $M \times N$. For which K, L, M, N the grid $M \times N$ has smaller expectation value than the grid $K \times L$?

9. (Quarks confinement in 2-dimensional space.) Is the amount of energy $E_M(N)$ required to move two quarks far away from each other finite or infinite?

10. Investigate 3- and 4-dimensional grids experimentally by a numeric simulation. Is the amount of energy required to move two quarks far away from each other finite or infinite? And if $2^{-S[U]}$ is replaced by $c^{-S[U]}$, where $c \in [2; 3]$, in the definition of the model?

A short intro to permutations here.

Non-Abelian case. The *trace* of a permutation $x \in S_3$ is $\mathrm{Tr}(x) = \begin{cases} 3, & \text{if } x \text{ is the identity;} \\ -1, & \text{if } x \text{ is a transposition;} \\ 0, & \text{if } x \text{ is a cycle of length 3.} \end{cases}$

Let the collection of rates $U(ab) \in S_3$ be random with the probability of a collection proportional to $2^{-S[U]}$, where $L(x) = 3 - \mathrm{Tr}(x)$. Let W be the trace of the product of all the rates $U(ab)$ for a counterclockwise travel around the whole boundary.

11. Compute the expectation of W. Prove that it is of order const^{MN}.

12. (The essence of the Millenium problem ???) The same for the 4-dimensional grid.