1. CATEGORIES AND HOMOLOGY

Problem 1. Prove that any functor $T : \mathcal{C} \to \mathcal{D}$ assigns isomorphic objects to isomorphic objects.

Problem 2. In an arbitrary category define the notion of right inverse and left inverse of a morphism $f : X \to Y$ and show that if they both exist, then they coincide and f is an isomorphism. Prove that any isomorphism has a unique inverse.

Problem 3. Let A and B be isomorphic objects in a category T. Prove that there exists a functor $\mathrm{Exch}_{A,B}$ from T to itself mapping A to B, B to A, and every other object ot T to itself.

A *groupoid* is a category where every morphism has an inverse. A path groupoid of a topological space X is a category where objects are points of X, morphisms between a and b are classes of homotopy of continuous paths in X joining a and b. Composition is the composition of paths.

Problem 4. Prove that path groupoids of homotopy equivalent spaces are equivalent categories.

Problem 5. A is called an initial (resp., terminal) object of a category T if for every object B of T there is exactly one morphism $A \to B$ (resp., $B \to A$). Prove that an initial object need not exist; may be non-unique, but any two initial objects are isomorphic.

Problem 6. Find initial and terminal objects in the following categories: (a) a category of subgroups of a given group (morphisms are homomorphisms); (b) a category of sets (morphisms are maps); (c) a category of coverings with a given base (morphisms are maps $f : E_1 \to E_2$ such that $p_2 \circ f = p_1$).

Problem 7. Prove that the following notions can be described as initial or terminal objects of certain categories, and describe these categories: (a) a free group with a given system of generators; (b) a tensor product of vector spaces; (c) a free product of two groups; (d) a wedge sum of topological spaces.

Suppose the following homologies to be known:

- $H_0(D^n) = \mathbb{Z}$, $H_i(D^n) = 0$ for $i > 0$.
- $H_0(S^n) = H_n(S^n) = \mathbb{Z}$, $H_i(S^n) = 0$ for all other i.
- $H_i(\mathbb{C}P^n) = \mathbb{Z}$ for $i = 0, 2, 4, \dots, 2n$; else $H_i(\mathbb{C}P^n) = 0$.
- $H_0(\mathbb{T}^2) = \mathbb{Z}$, $H_1(\mathbb{T}^2) = \mathbb{Z}^2$, $H_2(\mathbb{T}^2) = \mathbb{Z}$, $H_i(\mathbb{T}^2) = 0$ for all other i.
- $H_0(\mathbb{R}P^n) = \mathbb{Z}$, $H_n(\mathbb{R}P^n) = \mathbb{Z}$ if n is odd, $H_i(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ if $i < n$ and i is odd, $H_i(\mathbb{R}P^n) = 0$ in all the other cases.

Homology is a functor from the homotopy category to the category of abelian groups.

Problem 8. (a) Can the torus be retracted on its meridional circle? (b) Is there a retraction of S^5 on S^4, where S^4 is the "equator" of S^5? (c) Is there a retraction of $\mathbb{R}P^5$ on $\mathbb{R}P^4$, where $\mathbb{R}P^4 \hookrightarrow \mathbb{R}P^5$ is the natural embedding given by $[x^1 : \dots : x^5] \mapsto [x^1 : \dots : x^5 : 0]$?

Problem 9. (a) Is there a retraction of a (b) disk D^n (c) a solid torus $S^1 \times D^2$ on its boundary?

Problem 10. Given a continuous map $p : \mathbb{R}P^2 \to S^1 \times D^{17}$ and a homeomorphism $h : S^1 \times D^{17} \to S^1 \times D^{17}$, can h be lifted to a map $H : S^1 \times D^{17} \to \mathbb{R}P^2$ such that $p \circ H = h$?

Problem 11. (a) Prove that Euclidean spaces of different dimensions are not homeomorphic. (b) Prove that if $n \neq m$ then D^n is not homeomorphis to D^m.