6. MAYER–VIETORIS SEQUENCE

The following is for the students who solved Problem 4.2 (the Bockstein's exact sequence) or are ready to believe its statement.

Let $X = A \cup B$ be a CW-space, $A, B \subset X$ be its CW-subspaces (unions of cells that are CW-spaces) such that $A \cap B$ is a CW-subspace of both A and B. Let $u_A : A \to X$, $u_B : B \to X$, $w_A : A \cap B \to A$ and $w_B : A \cap B \to B$ be tautological embeddings. Denote by $Z(Y)$ the cell complex of a CW-space Y.

Problem 1. Prove that $0 \to Z(A \cap B) \xrightarrow{w_A \oplus (-w_B)} Z(A) \oplus Z(B) \xrightarrow{u_A + u_B} Z(A \cup B) \to 0$ is an exact sequence of complexes. The corresponding exact sequence of cell homology (see Problem 4.2) is called the Mayer–Vietoris sequence.

The Mayer–Vietoris sequence for singular homology is a bit more tricky: fix a barycentric subdivision of the standard simplex: $\Delta_n = \bigcup_{\sigma \in \Sigma_{n+1}} \Delta_{n,\sigma}$; here Σ_{n+1} is the permutation group of $0, 1, \dots, n$, and $\Delta_{n,\sigma} \stackrel{\text{def}}{=} \{(x_0, \dots, x_n) \in \Delta_n \mid x_{\sigma(0)} \le \dots \le x_{\sigma(n)}\}$. For every σ fix an affine map $w_{n,\sigma} : \Delta_n \to \Delta_{n,\sigma}$, sending the vertex $x_i = 1$ of Δ_n (where $i = 0, \dots, n$) to the vertex of $\Delta_{n,\sigma}$ where $x_j = 1/(i+1)$ if $j \in \{\sigma(0), \dots, \sigma(i)\}$, and $x_j = 0$ otherwise. Define the homomorphism $\beta_n : C_n(X) \to C_n(X)$ by $\beta_n(f) = \sum_{\sigma \in \Sigma_{n+1}} (-1)^{\text{sign}(\sigma)} f \circ w_\sigma$.

Problem 2*. Prove that β_n is a morphism of complexes, and $(\beta_n)_* : H_n(X) \to H_n(X)$ is trivial (here $H_n(X)$ is the singular homology of X).

Let $X = A \cup B$ where $A, B \subset X$ are open; denote by $C_n^{A,B}(X)$ the subcomplex of $C_n(X)$ spanned by singular simplices $f : \Delta_n \to X$ where $f(\Delta_n) \subset A$ or $f(\Delta_n) \subset B$.

Problem 3*. (a) Prove that the inclusion $\iota : C^{A,B}(X) \to C(X)$ is a morphism of complexes, and ι_* is trivial on homology — hence, the homology of $C^{A,B}(X)$ is the same as the singular homology of X. (b) Let $w_A : C(A \cap B) \to C(A)$, $w_B : C(A \cap B) \to C(B)$, $u_A : C(A) \to C^{A,B}(X)$ and $u_B : C(B) \to C^{A,B}(X)$ be tautological inclusions. Prove that $0 \to C(A \cap B) \xrightarrow{w_A \oplus (-w_B)} C(A) \oplus C(B) \xrightarrow{u_A + u_B} C^{A,B}(X) \to 0$ is an exact sequence of complexes. The corresponding exact sequence of singular homology is called the Mayer–Vietoris sequence.

Problem 4. Compute the Mayer–Vietoris sequence with (a) $X = S^n$, A and B are the upper and the lower half-sphere, respectively. (b) A and B are two cylinders glued together by the bases to form $X = \mathbb{T}^2$. (c) A and B are two cylinders glued together by the bases to form the Klein bottle.

Problem 5. Let $X = \Sigma Y$ be the suspension. Use the Mayer–Vietoris sequence to express $H_*(X)$ via $H_*(Y)$.

Problem 6. Let $X = S^3$, $K \subset S^3$ be a knot, A be its thin tubular neighbourhood, and B be the closure of $X \setminus A$. Compute the Mayer–Vietoris sequence of (X, A, B) if K is (a) an unknot, (b) a trefoil.

Problem 7. Look in Wikipedia (or elsewhere) when Leopold Vietoris was born, and when he died.