8. APPLICATIONS OF HOMOLOGY.

1. Fundamental classes

One says that a compact oriented submanifold $N \subset M$ of dimension n realizes a homology class $\iota_*(\tau_N) \in H_n(M)$ where $\iota : N \to M$ is the tautological embedding and $\tau_N \in H_n(N)$ is the fundamental class of N chosen according to the orientation. Similarly, any compact submanifold N realizes a homology class mod2 (here N may be not oriented and even non-orientable).

Problem 1. (a) Let $N \subset \mathbb{C}P^2$ be a smooth curve of degree n. Prove that N is orientable and realizes a class $n \in \mathbb{Z} = H_2(\mathbb{C}P^2)$. (b) Prove a similar statement for $\mathbb{R}P^2$. (c) Prove a similar statement for $H_{2n-2}(\mathbb{C}P^n)$. (d) Realize all the classes in $H_*(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$ by smooth submanifolds. Which of them realize classes in $H_*(\mathbb{R}P^n, \mathbb{Z})$?

2. Degree of a smooth map

Problem 2. (a) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible. Prove that the map $\widehat{A} = A|_{S_1^{n-1}} : S_1^{n-1} \to A(S_1^{n-1})$, where $S_1^{n-1} \subset \mathbb{R}^n$ is the unit sphere centered at the origin, is a diffeomorphism, and that $\widehat{A}_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is the multiplication by $\pm 1 = \operatorname{sign}\det A$. (b) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map such that $f(0) = 0$ and $f'(0)$ is nondegenerate. Prove that for $\varepsilon > 0$ small enough the map $f|_{S_\varepsilon^{n-1}} \to f(S_\varepsilon^{n-1})$, where $S_\varepsilon^{n-1} \subset \mathbb{R}^n$ is the ε-sphere centered in x, is a diffeomorphism homotopic to $\widehat{f'(0)}$.

Problem 3. Let M be a smooth n-manifold, and $U \subset M$ an open set diffeomorphic to an n-ball. Prove that $H_i(M) = H_i(M \setminus U, \partial U)$ for all i.

Let M, N be smooth manifolds of the same dimension n, $f : M \to N$ be a smooth map, and y be its regular value: if $f(x) = y$ then $f'(x) : T_x M \to T_y N$ is nondegenerate.

Problem 4. (a) Prove that $f^{-1} \subset M$ is discrete; if M is compact then it is finite: $f^{-1}(y) = \{x_1, \dots, x_N\}$. (b) Let $U_\varepsilon \subset N$ be an open ε-ball (in some Riemannian metric) centered in y. Prove that for ε small enough the preimage $f^{-1}(U_\varepsilon)$ is a finite disjoint union $\bigsqcup_{i=1}^N V_i$ where for every i one has $x_i \in V_i$ and the restriction $f|_{\partial V_i} : \partial V_i \to \partial U_\varepsilon$ is a diffeomorphism homotopic to $\widehat{f'(x_i)}$.

Problem 5. In the notation of Problem 4 consider the diagram

$$\begin{array}{ccc} H_n(M) = H_n(M \setminus \bigsqcup_i V_i, \bigsqcup_i \partial V_i) & \longrightarrow & H_{n-1}(\bigsqcup_i \partial V_i) \\ \downarrow f_* & & \downarrow f_* \\ H_n(N) = H_n(N \setminus U_\varepsilon, \partial U_\varepsilon) & \longrightarrow & H_{n-1}(\partial U_\varepsilon) \end{array}$$

where the horizontal arrows are part of the exact sequence of the pairs $(M \setminus \bigsqcup_i V_i, \bigsqcup_i \partial V_i)$ and $(N \setminus U_\varepsilon, \partial U_\varepsilon)$, respectively, and prove that $\deg f = \sum_{i=1}^N \operatorname{sign}\det f'(x_i)$.

3. Euler characteristic

Problem 6. Using the Mayer–Vietoris sequence, prove that $\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2)$, where X_1 and X_2 are simplicial subspaces of a simplicial space X. Prove the same equality "by counting simplices".

Problem 7. Compute the Euler characteristics of (a) $\mathbb{R}P^n$; (b) $\mathbb{C}P^n$; (c) all compact 2-manifolds.

Problem 8. Prove that the Euler characteristic of a compact smooth manifold of odd dimension is 0.

Problem 9. Prove that if $p : E \to B$ is a fiber bundle with a fiber F, and B, E and F are simplicial spaces then $\chi(E) = \chi(B)\chi(F)$. In particular, $\chi(B \times F) = \chi(B)\chi(F)$ and $\chi(E) = n\chi(B)$ if p is an n-sheeted covering.

Problem 10. Let $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ be a meromorphic function of degree n, and $a_1, \dots, a_k$ be its critical points. Call $d_i \in \mathbb{Z}_{\geq 0}$ the multiplicity of f at a_i if $f(z) - f(a_i) = (z - a_i)^{d_i} + o((z - a_i)^{d_i})$; here z is any local holomorphic coordinate on $\mathbb{C}P^1$ near a_i. Prove that $\sum_{i=1}^k d_i = 2n - 2 + k$. What does this formula give if f is a polynomial?

Problem 11. Let $M \subset \mathbb{R}^n$ be a compact oriented hypersurface (smooth submanifold of dimension $n-1$). For $x \in M$ denote by $v(x)$ the unit vector normal to M at x; the choice between two such vectors is dictated (how?) by the orientation of M. So, v is the map $M \to S^{n-1}$. Prove that if n is even then $\deg v = 0$, otherwise $\deg v = 2\chi(M)$.

4. Fixed points

Problem 12. Construct a continuous map $f : X \to X$ without fixed points or prove that it does not exist. Can f be homotopic to the identity map? (a) $X = S^n$; (b) $X = \mathbb{R}P^n$; (c) $X = \mathbb{C}P^n$; (d) X is a sphere with g handles; (e) X is a 2-disk with n holes, $n > 0$.