Lecture 1

The topology of subsets of $\mathbb{R}^n$

The basic material of this lecture should be familiar to you from Advanced Calculus courses, but we shall revise it in detail to ensure that you are comfortable with its main notions (the notions of open set and continuous map) and know how to work with them.

1.1. Continuous maps

"Topology is the mathematics of continuity"

Let $\mathbb{R}$ be the set of real numbers. A function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ is called *continuous at the point* $x_0 \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that the inequality

$$|f(x_0) - f(x)| < \varepsilon$$

holds for all $x \in \mathbb{R}$ whenever $|x_0 - x| < \delta$. The function f is called *continuous* if it is continuous at all points $x \in \mathbb{R}$.

This is basic one-variable calculus.

Let $\mathbb{R}^n$ be n-dimensional space. By $O_r(p)$ denote the *open ball* of radius $r > 0$ and center $p \in \mathbb{R}^n$, i.e., the set

$$O_r(p) := \{q \in \mathbb{R}^n : d(p, q) < r\},$$

where d is the distance in $\mathbb{R}^n$. A function $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is called *continuous at the point* $p_0 \in \mathbb{R}^n$ if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $f(p) \in O_\varepsilon(f(p_0))$ for all $p \in O_\delta(p_0)$. The function f is called *continuous* if it is continuous at all points $p \in \mathbb{R}^n$.

This is (more advanced) calculus in several variables.

A set $G \subset \mathbb{R}^n$ is called *open in* $\mathbb{R}^n$ if for any point $g \in G$ there exists a $\delta > 0$ such that $O_\delta(g) \subset G$. Let $X \subset \mathbb{R}^n$. A subset $U \subset X$ is called *open in* X if for any point $u \in U$ there exists a $\delta > 0$ such that $O_\delta(u) \cap X \subset U$. An equivalent property: $U = V \cap X$, where V is an open set in $\mathbb{R}^n$. Clearly, any union of open sets is open and any finite intersection of open sets is open. Let X and Y be subsets of $\mathbb{R}^n$. A map $f\colon X \longrightarrow Y$ is called *continuous* if the preimage of any open set is an open set, i.e.,

$$V \text{ is open in } Y \implies f^{-1}(V) \text{ is open in } X.$$

This is basic topology.

Let us compare the three definitions of continuity. Clearly, the topological definition is not only the shortest, but is conceptually the simplest. Also, the topological definition yields the simplest proofs. Here is an example.

Theorem 1.1. *The composition of continuous maps is a continuous map. In more detail, if X, Y, Z are subsets of $\mathbb{R}^n$, $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are continuous maps, then their composition, i.e., the map $h = g \circ f\colon X \longrightarrow Z$ given by $h(x) := g(f(x))$, is continuous.*

Proof. Let $W \subset Z$ be open. Then the set $V := f^{-1}(W) \subset Y$ is open (because f is continuous). Therefore, the set $U := g^{-1}(V) \subset X$ is open (because g is continuous). But $U = h^{-1}(W)$. □

Compare this proof with the proof of the corresponding theorem in basic calculus. This proof is much simpler.

The notion of open set, used to define continuity, is fundamental in topology. Other basic notions (neighborhood, closed set, closure, interior, boundary, compactness, path connectedness, etc.) are defined by using open sets.

1.2. Closure, boundary, interior

By a *neighborhood* of a point $x \in X \subset \mathbb{R}^n$ we mean any open set (in X) that contains x.

Let $A \subset X$; an *interior point* of A is a point $x \in A$ which has a neighborhood U in X contained in A. The set of all interior points of A is called the *interior* of A in X and is denoted by $\mathrm{Int}(A)$. An *isolated point* of A in X is a point $a \in A$ which has a neighborhood U in X such that $U \cap A = a$.

A *boundary point* of A in X is a point $x \in X$ such that any neighborhood $U \ni x$ in X contains points of A and points not in A, i.e., $U \cap A \neq \varnothing$ and $U \cap (X - A) \neq \varnothing$; the boundary of A is denoted by $\mathrm{Bd}(A)$ or ∂A. The union of A and all the boundary points of A is called the *closure* of A in

X and is denoted by $\mathrm{Clos}(A, X)$ (or $\mathrm{Clos}(A)$, or $\overline{A}$, if X is clear from the context).

Theorem 1.2. *Let $A \subset \mathbb{R}^n$.*

(a) *A is closed if and only if it contains all of its boundary points.*

(b) *The interior of A is the largest (by inclusion) open set contained in A.*

(c) *The closure of A is the smallest (by inclusion) closed set containing A.*

(d) *The boundary of a set A is the difference between the closure of A and the interior of A:* $\mathrm{Bd}(A) = \mathrm{Clos}(A) - \mathrm{Int}(A)$.

The proofs follow directly from the definitions, and you should remember them from the Calculus course. You should be able to write them up without much trouble in the exercise class.

1.3. Topological equivalence

"A topologist is person who can't tell the difference between a coffee cup and a doughnut."

The goal of this section is to teach you to visualize objects (geometric figures) the way topologists see them, i.e., by regarding figures as equivalent if they can be bijectively deformed into each other. This is something you have not been taught to do in calculus courses, and it may take you some time before you will become able to do it.

Let X and Y be "geometric figures," i.e., arbitrary subsets of $\mathbb{R}^n$. Then X and Y are called *topologically equivalent* or *homeomorphic* if there exists a *homeomorphism* of X onto Y, i.e., a continuous bijective map $h\colon X \longrightarrow Y$ such that the inverse map h^{-1} is continuous.

For the topologist, homeomorphic figures are the same figure: a circle is the same as the boundary of a square, or that of a triangle, of a hexagon, of an ellipse; an arc of a circle is the same as a closed interval, a 2-dimensional disk is the same as the square, or as a triangle together with its inner points; the boundary of a cube is the same as a sphere, or as the boundary of a cylinder, or (the boundary of) a tetrahedron.

If a property does not change under any homeomorphism, then this property is called *topological.* Examples of topological properties are compactness and path connectedness (they will be defined later in this lecture). Examples of properties that are *not* topological are length, area, volume, and boundedness. The fact that boundedness is not a topological property may seem rather surprising; as an illustration, we shall prove that

the open interval $(0, 1)$ *is homeomorphic to the real line* $\mathbb{R}$ (!)

This is proved by constructing an explicit homeomorphism $h\colon (0,1) \longrightarrow \mathbb{R}$ as the composition of the two homeomorphisms p and s shown in Figure 1.1.

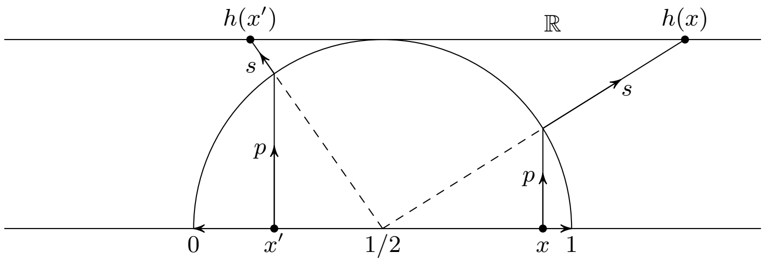


FIGURE 1.1. The homeomorphism $h\colon (0,1) \longrightarrow \mathbb{R}$

For another illustration, look at Figure 1.2; you should intuitively feel that the torus is not homeomorphic to the sphere (although we are at present unable to prove this!). However, the ordinary torus *is* homeomorphic to the knotted torus in the figure, although they look "topologically very different"; they provide examples of figures that are homeomorphic, but are embedded in $\mathbb{R}^3$ in different ways. We shall come back to this distinction later in the course, in particular in the lecture on knot theory.

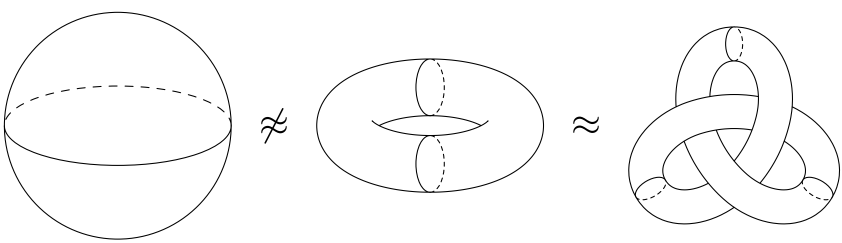


FIGURE 1.2. The sphere and two tori

We conclude this lecture by studying two basic topological properties of geometric figures that will be constantly used in this course.

1.4. Path connectedness

A set $X \subset \mathbb{R}^n$ is called *path connected* if any two points of X can be joined by a path, i.e., if for any $x,\ y \in X$ there exists a continuous map $\varphi\colon [0,1] \longrightarrow X$ such that $\varphi(0)=x$ and $\varphi(1)=y$.

Theorem 1.3. *The continuous image of a path connected set is path connected. In more detail, if the map $f\colon X \longrightarrow Y$ is continuous and X is path connected, then $f(X)$ is path connected.*

Proof. Let $y_1, y_2 \in f(X)$. Let $X_1 := f^{-1}(y_1)$ and $X_2 := f^{-1}(y_2)$. Let x_1 and x_2 be arbitrary points of X_1 and X_2, respectively. Then there exists a continuous map $\varphi\colon [0,1] \longrightarrow X$ such that $\varphi(0)=x_1$ and $\varphi(1)=x_2$ (because X is path connected). Let $\psi\colon [0,1] \longrightarrow f(X)$ be defined by $\psi := f \circ \varphi$. Then ψ is continuous (by Theorem 1.1), $\psi(0)=y_1$ and $\psi(1)=y_2$. □

Thus we have shown that path connectedness is a topological property.

1.5. Compactness

A family $\{U_\alpha\}$ of open sets in $X \subset \mathbb{R}^n$ is called an *open cover* of X if this family covers X, i.e., if $\bigcup_\alpha U_\alpha \supset X$. A *subcover* of $\{U_\alpha\}$ is a subfamily $\{U_{\alpha_\beta}\}$ such that $\bigcup_\beta U_{\alpha_\beta} \supset X$, i.e., the subfamily also covers X. The set X is called *compact* if every open cover of X contains a finite subcover.

Note the importance of the word "every" in the last definition: a set in noncompact if *at least one* of its open covers contains no finite subcover of X. As an illustration, let us show that

the open interval $(0,1)$ *is not compact.*

Indeed, this follows from the fact that any finite subfamily of the cover $\{U_1, U_2, \ldots\}$ shown in Figure 1.3 obviously does not cover $(0,1)$.

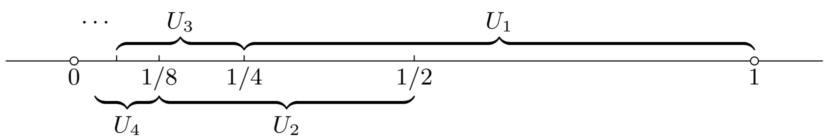


FIGURE 1.3. The open interval $(0,1)$ is not compact

Theorem 1.4. *The continuous image of a compact set is compact, i.e., if a map $f\colon X \longrightarrow Y$ is continuous and $X \subset \mathbb{R}^n$ is compact, then $f(X)$ is compact.*

Proof. Let $\{V_\alpha\}$ be an open covering of $f(X)$. Then each $U_\alpha := f^{-1}(V_\alpha)$ is open in X (by the definition of continuity) and so $\{U_\alpha\}$ is an open covering of X. But X is compact, hence $\{U_\alpha\}$ has a finite subcovering, say $\{U_{\alpha_1}, \dots, U_{\alpha_N}\}$. Then $\{f(U_{\alpha_1}), \dots, f(U_{\alpha_N})\}$ is obviously a finite subcover of $\{V_\alpha\}$. □

Thus we have shown that compactness is a topological property.

Fact 1.5. *A set $X \subset \mathbb{R}^n$ is compact if and only if X is closed and bounded.*

We do not give the proof of this fact because it not really topological: the word "bounded" makes no sense to a topologist; the proof is usually given in calculus courses.

1.6. Exercises

1.1. Using the ε-δ definition of continuity, give a detailed proof of the fact that the composition of two continuous functions is continuous.

1.2. Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$. Suppose the functions $f_{1,x_0}(y) := F(x_0, y)$ and $f_{2,y_0}(x) := F(x, y_0)$ are continues for any $x_0, y_0 \in \mathbb{R}$. Is it true that $F(x, y)$ is continuous?

1.3. Prove the four assertions (a)–(d) of Theorem 1.2.

1.4. The towns A and B are connected by two roads. Two travellers can walk along these roads from A to B so that the distance between them at any moment is less than or equal to 1 km. Can one traveller walk from A to B and the other from B to A (using these roads) so that the distance between them at any moment is greater than 1 km?

1.5. Suppose $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. The *distance* from the point x to the subset A is equal to $d(x, A) = \inf\{\|x - a\| : a \in A\}$.

(i) Prove that the function $f(x) = d(x, A)$ is continuous for any $A \subset \mathbb{R}^n$.

(ii) Prove that if the set A is closed, then the function $f(x) = d(x, A)$ is positive for any $x \notin A$.

1.6. Let X be the subset of $\mathbb{R}^2$ given by the equation $xy = 0$ (X is the union of two lines). Give some examples of neighborhoods: (a) of the point $(0, 0)$; (b) of the point $(0, 1)$.

1.7. Describe the set of points x in $\mathbb{R}^2$ such that $d(x, A) = 1;\ 2;\ 3$, where the set A is given by the formula:

(a) $x^2 + y^2 = 0$; (b) $x^2 + y^2 = 2$;

(c)* $x^2 + 2y^2 = 2$; (d) the square of area two.

1.8. Let A and B be two subsets of the set X that was defined in Exercise 1.6. Suppose that A and B are homeomorphic and A is open in X. Is it true that B is also open in X?

1.9. Construct a homeomorphism between the boundary of the cube $\mathbb{I}^3$ and the sphere $\mathbb{S}^2$.

1.10. Construct a homeomorphism between the plane $\mathbb{R}^2$ and the open disk $\mathbb{B}^2 := \{v \in \mathbb{R}^2 : |v| < 1\}$.

1.11. Construct a homeomorphism between the plane $\mathbb{R}^2$ and the sphere $\mathbb{S}^2$ with one point removed.

Lecture 2

Abstract topological spaces

In this lecture, we move from the topological study of concrete geometrical figures (subsets of $\mathbb{R}^n$) to the axiomatic study of abstract topological spaces. What is remarkable about this approach is the simplicity of the underlying axioms (based on the notion of open set, now an undefined concept in the axiomatics), which nevertheless allow to generalize the deep theorems about subsets of $\mathbb{R}^n$ (proved in the previous lecture) to subsets of any abstract topological space, by reproducing the proofs practically word for word.

2.1. Topological spaces

By definition, an (*abstract*) *topological space* $(X, \mathcal{T} = \{U_\alpha\})$ is a set X of arbitrary elements $x \in X$ (called *points*) and a family $\mathcal{T} = \{U_\alpha\}$ (called the *topology of the space* X) of subsets of X (called *open sets*) such that

(1) X and $\varnothing$ are open;

(2) if U and V are open, then $U \cap V$ is open;

(3) if $\{V_\beta\}$ is any collection of open sets, then the set $\bigcup_\beta V_\beta$ is open.

Any set $X \subset \mathbb{R}^n$ is a topological space if the family of open sets is defined as in Section 1.1. (The proof is a straightforward exercise.) All the definitions from Sections 1.2–1.4 are valid for any topological space (and not only for subsets of $\mathbb{R}^n$), because they only use the notion of open set. All the theorems (and their proofs) from the previous lecture are also valid. At this point the reader should read through these proofs again and check that, indeed, only the properties of open sets appearing in the axioms are used.

In order to define a topological space, we don't have to specify *all* the open sets: there is a more "economical" way of defining the topology. For a topological space $(X, \mathcal{T})$, we say that a subset $\mathcal{T}_0 \subset \mathcal{T} = \{U_\alpha\}$) is a *base* *of the topology* of $(X, \mathcal{T})$ if for any open set $U \in \mathcal{T}$ there exists a collection $\{V_\beta\}$ of open sets in $\mathcal{T}_0$ such that $U = \bigcup_\beta V_\beta$.

Clearly, any base of the topology uniquely determines the whole topology (how?). For example, the set of all open balls in $\mathbb{R}^n$ is a base of the standard topology of Euclidean space.

Examples. (1) Any set D becomes a topological space if it is supplied with the *discrete topology*, i.e., if any set is declared open. Obviously, a topology is discrete if and only if any point is an open set.

(2) Any set X supplied with only two open sets (the empty set and X itself) is a topological space with the *trivial topology.*

(3) Any metric space M (see the definition in the next section) is a topological space in the *metric topology*, which is given by the base of all open balls $O_r(m) := \{m' \colon d(m', m) < r\}$ in M, where d is the distance function in M.

(4) The space $\mathcal{C}[0, 1]$ of continuous real-valued functions on the closed interval $[0, 1] \subset \mathbb{R}$ has a standard topology given by the base of open balls $O_r(f) := \{g \colon \sup_x(|g(x) - f(x)|) < r\}$.

Many more nontrivial examples will be given at the end of this lecture, in the exercise class and in subsequent lectures.

2.2. Metric spaces

A *metric space* is a set M supplied with a *metric* (or *distance function*), i.e., a function $d\colon M \times M \longrightarrow \mathbb{R}$ such that

(1) for all $x, y \in M$, $d(x, y) \geqslant 0$ (*nonnegativity*);
(2) for all $x, y \in M$, $d(x, y) = 0$ iff $x = y$; (*identity*);
(3) for all $x, y \in M$, $d(x, y) = d(y, x)$ (*symmetry*);
(4) for all $x, y, z \in M$, $d(x, z) \leqslant d(x, y) + d(y, z)$ (*triangle inequality*).

The most popular example of a metric space is Euclidean space $\mathbb{R}^n$ (and its subsets) with the standard metric:

$$d(p, q) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}, \quad \text{where } p = (x_1, \dots, x_n),\ q = (y_1, \dots, y_n).$$

Other less familiar examples will appear in the exercise classes.

As we mentioned above, any metric space (M, d) becomes a topological space in the metric topology. Conversely, it is *not* true that any topological space $(X, \mathcal{T})$ has a metric (i.e., possesses a distance function for which the metric topology coincides with $\mathcal{T}$). Until the middle of the 20th century one of the main problems of topology was to find necessary and sufficient conditions for a topological space $(X, \mathcal{T})$ to be *metrizable*, i.e., for X to have a metric such that the corresponding metric topology coincides with $\mathcal{T}$.

2.3. Induced topology

If A is a subset of a topological space X, then A acquires a topological structure in a natural way: the topology on A is *induced* from X if we declare all the intersections of open sets of X with A to be the open sets of A. It is easy to check that A with the induced topology is indeed a topological space (i.e., satisfies axioms (1)–(3) from Section 2.1).

It is important to note that open sets in the induced topology of A are not necessarily open in X (in fact, in most cases they are not).

Whenever we consider a subset of a topological space, we will always regard it as a topological space in the induced topology without explicit mention. Speaking of open sets, however, one should always make clear with respect to what set or subset openness is understood. Thus the open interval (0, 1) is open on the real line, but not in the plane.

2.4. Connectedness

In the previous lecture, we defined path connectedness of subsets of $\mathbb{R}^n$; that definition remains valid, word for word, for topological spaces. Intuitively, pathconnectedness of a topological space means that you can move continuously within the space from any point to any other point. But there is another definition of connectedness based on the idea that a connected set is "a set that consists of one piece". The rigorous formalization of the idea of "consisting of one piece" is as follows.

A topological space X is called *connected* if it is not the union of two open, closed, nonempty, and nonintersecting sets, i.e., $X = A \cup B$, where A and B are both open, closed, and nonempty, implies $A \cap B \neq \varnothing$.

What is the relationship between the notions of connectedness and path connectedness?

Theorem 2.1. *Any path connected topological space is connected, but there exist connected topological spaces that are not path connected.*

Proof. Suppose that the space X is path connected. Arguing by contradiction, let us assume that it is the disjoint union of two open and closed nonempty sets A and B. Let $a \in A$, $b \in B$. Then there exists a continuous map $f\colon [0,1] \longrightarrow X$ such that $f(0)=a$ and $f(1)=b$. Denote $A_0 := f^{-1}(A)$ and $B_0 := f^{-1}(B)$. These two sets are disjoint, open (as inverse images of open sets) and cover the closed interval $[0,1]$ (because $f([0,1]) \subset X = A \cup B$). We know that $1 \in B_0$. Let ξ be the least upper bound of A_0. If $\xi \in A_0$, then A_0 cannot be open, so ξ belongs to B_0; but then B_0 cannot be open. A contradiction.

Concerning the converse statement, see Exercise 2.12. □

Connectedness, like path connectedness, is not only a topological property—it is preserved by *any* continuous maps (not only by homeomorphisms).

Theorem 2.2. *The continuous image of a connected set is connected, i.e., if a map $f\colon X \longrightarrow Y$ is continuous and X is connected, then $f(X)$ is connected.*

Proof. We argue by contradiction: suppose that X is connected, but $f(X)$ is not. Then $f(X) = A \cup B$, where both A and B are both closed and open, and don't intersect. Denote $A' = f^{-1}(A)$ and $B' = f^{-1}(B)$. Then $X = A \cup B$, $A \cap B = \varnothing$, both A and B are open (as preimages of open sets) and closed (as complements to open sets). But this means that X is not connected—a contradiction. □

Roughly speaking, a connected component of a nonconnected set is just one of its many "pieces". The formal definition is this: a *connected component* of a not necessarily connected space X is any connected subset of X not contained in a larger connected subset of X. It is easy to prove that any connected component of a space X is both open and closed in X.

2.5. Separability

An important type of property for topological spaces comes from various separability axioms, which specify how well it is possible to "separate" points and/or sets (i.e., put them into nonintersecting neighborhoods). We only define one such property, the most natural and classical one: a topological space is said to be a *Hausdorff space* if any two distinct points possess nonintersecting neighborhoods. Obviously, Euclidean space and any of its subsets are Hausdorff, as are indeed any metric spaces (why?). The sad fact that there exist non-Hausdorff spaces will be considered in the exercise class.

2.6. More examples of topological spaces

In this section, we list twelve classical mathematical objects (not necessarily familiar to you) coming from completely different areas of mathematics. All of them are topological spaces. In the exercise class (and in doing the homework assignments), you will learn how to define their topology (by introducing an appropriate base). You will perhaps be surprised to learn that certain objects from different parts of mathematics and physics, which at first glance have nothing in common, turn out to be topologically equivalent (homeomorphic).

We begin with examples coming from algebra.

(1) The group $\mathrm{Mat}(n, n)$ of all nondegenerate $n \times n$ matrices.

(2) The group $\mathrm{O}(n)$ of all orthogonal transformations of $\mathbb{R}^n$.

(3) The set of all polynomials of degree n with leading coefficient 1.

The next examples come from geometry.

(4) The real projective space $\mathbb{R}P^n$ of dimension n.

(5) The Grassmanian $G(k, n)$, i.e., the set of k-dimensional planes containing the origin in n-dimensional affine space.

(6) The hyperbolic plane.

The next example comes from complex analysis.

(7) The Riemann sphere $\overline{\mathbb{C}}$ and, more generally, Riemann surfaces.

Here are some examples from classical mechanics.

(8) The configuration space of a solid rotating about a fixed point in 3-space.

(9) The configuration space of a rectilinear rod rotating in 3-space about (a) one of its extremities, (b) its midpoint.

Here are two from algebraic geometry.

(10) The set of solutions $p = (x_1, \dots, x_9) \in \mathbb{R}^9$ of the following system of 6 equations:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= 1, & x_1x_4 + x_2x_5 + x_3x_6 &= 0,\\ x_4^2 + x_5^2 + x_6^2 &= 1, & x_1x_7 + x_2x_8 + x_3x_9 &= 0,\\ x_7^2 + x_8^2 + x_9^2 &= 1, & x_4x_7 + x_5x_8 + x_6x_9 &= 0. \end{aligned}$$

(11) Any affine variety in the *Zariski topology* is a topological space.

In conclusion, an example from dynamical systems (differential equations).

(12) The phase space of billiards on the disk.

2.7. Exercises

2.1. Prove that any constant map is continuous.

2.2. For any subsets A, $B \subset \mathbb{R}^n$, define the *distance* between A and B by putting $d(A, B) := \inf\{\|a-b\| : a \in A, b \in B\}$.

(a) Is it true that $d(A, C) \leqslant d(A, B) + d(B, C)$?

(b) Let $A \subset \mathbb{R}^n$ be a closed subset, let $C \subset \mathbb{R}^n$ be a compact subset. Prove that there exists a point $c_0 \in C$ such that $d(A, C) = d(A, c_0)$. Further, prove that if the set A is also compact, then there exists a point $a_0 \in A$ such that $d(A, C) = d(a_0, c_0)$.

2.3. Prove that any closed subspace of a compact space is compact.

2.4. Prove that the topology of $\mathbb{R}^n$ has a countable base (i.e., a base consisting of a countable family of open sets).

2.5. Introduce a "natural" topology on

(a) the set $\mathrm{Mat}(m, n)$ of matrices of size $n \times m$;

(b) the real projective space $\mathbb{R}P(n)$ of dimension n;

(c) the Grassmannian $G(k, n)$, i.e., the set of k-dimensional planes containing the origin of n-dimensional affine space;

(d) the set of solutions $p = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ of the following system of two equations: $x_1^2 + x_2^3 + x_3^4 + x_4^5 = 1$ and $x_1x_2x_3x_4 = -1$;

(e) the set of all polynomials of degree n with leading coefficient 1.

2.6. (a) Is the topological space $\mathrm{GL}(n)$ connected?

(b) Prove that the topological space $\mathrm{SO}(3)$ is connected.

(c) Prove that the topological space $\mathrm{GL}(3)$ consists of two connected components.

2.7. (a) Prove that the function $d(x, y) = \max\{|x_i - y_i|, i = 1, \dots, n\}$ where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ defines a metric in $\mathbb{R}^n$.

(b) Prove that $d(x, y) = \sum_{i=1}^{n} |x_i - y_i|$ is a metric in $\mathbb{R}^n$.

(c) Draw some ε-neighborhood of the point $(0, 0, \dots, 0)$ in the metrics defined in (a) and (b).

2.8. Prove that any metric space is Hausdorf and construct an example of a non-Hausdorff space.

2.9. Let X be a Hausdorff space. Prove that for any two distinct points $x,\ y \in X$ there exists a neighborhood $U \ni x$ such that its closure does not contain the point y.

2.10. Let C be a compact subspace of a Hausdorff space X. Let $x \in X \setminus C$. Prove that the point x and the set C have disjoint neighborhoods.

2.11. Prove that any two disjoint compact subsets of a Hausdorff space have disjoint (open) neighborhoods.

2.12. Give an example of a connected topological space which is not path connected.

Lecture 3
Topological constructions

In this lecture, we study the basic constructions used in topology. These constructions transform one or several given topological spaces into a new topological space. Starting with the simplest topological spaces and using these constructions, we can create more and more complicated spaces, including those which are the main objects of study in topology.

3.1. Disjoint union

The *disjoint union* of two topological spaces X and Y, in the case when the two sets X and Y do not intersect, is the union of the sets X and Y with the following topology: a set W in $X \cup Y$ is open if the sets $W \cap X$ and $W \cap Y$ are open in X and Y, respectively; if the two sets X and Y intersect, the definition is a little trickier: first we artificially make them nonintersecting by considering, instead of the set Y, the same set of elements but marked, say, with a star, i.e., $Y^* := \{(y, *) \colon y \in Y\}$, and then proceed as before, declaring that a set W in $X \cup Y^*$ is open if the sets $W \cap X$ and $W \cap Y^*$ are open in X and Y^*, respectively. In both cases, we obtain a topological space denoted by $X \sqcup Y$.

This choice of topology ensures that both natural inclusions $X \hookrightarrow X \cup Y$ $(x \mapsto x)$ and $Y \hookrightarrow X \cup Y$ $(y \mapsto y)$ are continuous maps.

It is easy to see that the subsets X and Y (we do not explicitly write the stars (if any) in Y^*, but consider them implicitly present) are both open and closed in $X \sqcup Y$, so that the set $X \sqcup Y$ is not connected (provided both X and Y are nonempty).

3.2. Cartesian product

Roughly speaking, the Cartesian product of two spaces is obtained by putting a copy of one of the spaces at each point of the other space.

More precisely, let X and Y be topological spaces; consider the set of pairs $X \times Y = \{(x, y) \colon x \in X, y \in Y\}$ and make $X \times Y$ into a topological space by defining its base: a set $W \subset X \times Y$ belongs to the base if it has the form $W = U \times V$, where U is an open set in X and V is open in Y. It is easy to check that in this way we obtain a topological space, which is called the *Cartesian product* of the spaces X and Y.

This choice of topology ensures that both natural projections $X \times Y \longrightarrow X$ $((x, y) \mapsto x)$ and $X \times Y \longrightarrow Y$ $((x, y) \mapsto y)$ are continuous maps.

Classical examples: (i) the Cartesian product of two closed intervals is the square; (ii) the Cartesian product of two circles is the torus; (iii) the Cartesian product of two real lines $\mathbb{R}$ is the plane $\mathbb{R}^2$.

Theorem 3.1. *The Cartesian product of the n-disk and the m-disk is the $(n+m)$-disk. The Cartesian product of $\mathbb{R}^n$ and $\mathbb{R}^m$ is $\mathbb{R}^{n+m}$.*

The proof is absolutely straightforward.

3.3. Quotient spaces

Roughly speaking, a quotient space is obtained from a given space by identifying the points of certain subsets of the given space ("dividing" our space by these subsets).

More precisely let X be a topological space and let $\sim$ be an equivalence relation on the set X; we then consider the equivalence classes with respect to this relation as points of the quotient set $X/_\sim$ and introduce a topology in this set by declaring open any subset $U \subset X/_\sim$ such that $U^* := \{x \in \xi \colon \xi \in U\}$ is open in X. The topological space thus obtained is denoted by $X/_\sim$.

This choice of topology ensures that the natural projection $X \longrightarrow X/_\sim$ $(x \mapsto \xi_\beta$, where $\xi_\beta \ni x)$ is a continuous map.

Suppose X and Y are topological spaces, A and B are closed subspaces of X and Y, respectively, and $f \colon A \longrightarrow B$ is a continuous map. (The particular case in which f is a homeomorphism is often considered.) In the disjoint union of X and Y, we identify all points of each set in the family

$$\mathcal{F}_b := \{b \sqcup f^{-1}(b) \colon b \in B\}.$$

Then we denote the quotient space $(X \cup Y)/_\sim$, where $\sim$ is the equivalence relation identifying points in each of the sets $\mathcal{F}_b$, $b \in B$, $X \cup_f Y$ and say that this space is obtained by *attaching* (or *gluing*) Y to X along f.

If A is a subset of a topological space X, we denote by X/A the quotient space w. r. t. the equivalence relation $x \sim y$ iff $x, y \in A$. For example, we have $\mathbb{D}^n/\partial\mathbb{D}^n \approx \mathbb{S}^n$.

3.4. Cone, suspension, and join

(i) Roughly speaking, the cone over a space is obtained by joining a fixed point by line segments with all the points of the space. More precisely, let X be a topological space; consider the Cartesian product $X \times [0, 1]$ (called the *cylinder* over X) and on it, the equivalence relation $(x, 1) \sim (y, 1)$ for any $x, y \in X$; we define the *cone over* X as the quotient space of the cylinder by the equivalence relation $\sim$:

$$C(X) := (X \times [0, 1])/_{\sim}.$$

Note that all the points (x, t) with $t = 1$ are identified into one point, called the *vertex* of the cone. By definition, the cone over the empty set is one point. The cone over a point is a line segment, the cone over the circle is homeomorphic to the disk (although it is more natural to think of it as the lateral surface of the ordinary circular cone).

(ii) Roughly speaking, the suspension over a topological space is obtained by joining two fixed points by segments with all the points of the given space. Another heuristic way of saying this is that the suspension is a double cone (on “different sides”) over that space.

More precisely, let X be a topological space; consider the Cartesian product $X \times [-1, 1]$ and on it, the equivalence relation

$$(x, 1) \approx (y, 1) \quad \text{and} \quad (x, -1) \approx (y, -1)$$

for any $x, y \in X$; now define the *suspension over* X as the quotient space of the cylinder $X \times [-1, 1]$ by the equivalence relation $\approx$:

$$\Sigma(X) := (X \times [-1, 1])/_{\approx}.$$

By definition, the suspension over the empty set is the two point set $\mathbb{S}^0$. The suspension over the two point set is homeomorphic to the circle, that over the circle is homeomorphic to the 2-sphere.

The notion of suspension is extremely important in topology, particularly in algebraic topology (surprisingly, it is much more important than that of the cone).

(iii) Roughly speaking, the join of two spaces is obtained by joining each pair of points from the two spaces by a segment.

More precisely, suppose that X and Y are topological spaces; consider the Cartesianproduct $X \times [-1, 1] \times Y$ and identify (via an equivalence relation that will be denoted by $\equiv$) all pairs of points of the form $(x_1, 1, y) \equiv (x_2, 1, y)$ as well as all pairs of the form $(x, -1, y_1) \equiv (x, -1, y_2)$. The topological space $X * Y$ thus obtained,

$$X * Y := (X \times [-1, 1] \times Y)/_{\equiv},$$

is called the *join* of the spaces X and Y.

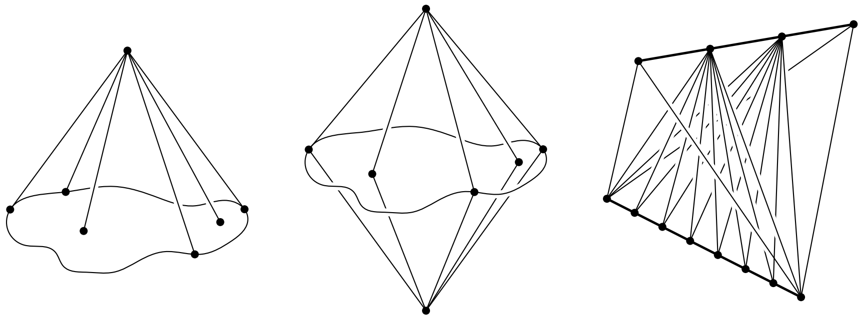


FIGURE 3.1. Cone and suspension. Join of two closed intervals

Theorem 3.2. *The cone over the n-sphere is the* $(n+1)$*-disk and the cone over the n-disk is the* $(n+1)$*-disk. The suspension over the n-sphere is the* $(n+1)$*-sphere and the suspension over the n-disk is the* $(n+1)$*-disk. The join of the n-disk and the m-disk is the* $(n+m+1)$*-disk. The join of the n-sphere and the m-sphere is the* $(n+m+1)$*-sphere.*

The proof is not difficult: one performs the construction in a Euclidean space of the appropriate dimension; in each case the corresponding homeomorphism is not hard to construct, although for large values of n and m it is difficult to visualize. The simplest (and only really "visual") nontrivial example is the join of two segments (which is the tetrahedron, otherwise known as the 3-simplex); it is shown in Figure $\overline{3}$.1.

3.5. Simplicial spaces

A 0-simplex is a point, a 1-simplex is a closed interval, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and so on. More generally and precisely, we define an *n-dimensional simplex* σ_n (*n-simplex* for short) as a topological space supplied with a homeomorphism

$$h\colon \sigma_n \longrightarrow \Delta^n = [e_0, e_1, \ldots, e_n],$$

where Δ^n is the convex hull of the set of $n+1$ points consisting of the origin $0=e_0$ and the endpoints $e_1, \dots, e_n$ of the basis unit vectors of Euclidean space $\mathbb{R}^n$. The n-simplex is of course homeomorphic to the n-disk $\mathbb{D}^n$, but it has a richer structure coming from the homeomorphism h. Namely, for any i, $0 \leqslant i \leqslant n$, it has a set of *i-faces*, each i-face is the preimage under h of the convex hull in $\mathbb{R}^n$ of i points from the set $\{e_0, e_1, \dots, e_n\}$. The 0-faces of an n-simplex are called *vertices*, and we often write

$$\sigma_n = [0, 1, \dots, n],$$

where by abuse of notation i, $i=0, 1, \dots, n$, denotes the vertex $h^{-1}(e_i)$.

Thus the 3-simplex possesses four 2-faces (triangles), six 1-faces (edges) and four 0-faces (vertices). By convention, we agree that the empty set is regarded as the (-1)-dimensional simplex. Note that the 3-simplex (as well as its faces), inherits a linear structure from $\mathbb{R}^3$ by the homeomorphism $h\colon \sigma_3 \longrightarrow \Delta^3 \subset \mathbb{R}^3$.

We now define a *finite simplicial space* X (also called *finite simplicial complex*) as the space obtained from the disjoint union of a finite set of simplices by gluing some of their faces together by homeomorphisms; it is assumed that the attaching homeomorphisms respect the linear structure of the faces (so that after the gluing is performed, all the simplices have a coherent linear structure). In this course, we will not consider the more general notion of simplicial space with a possibly infinite number of simplices, and so will often drop the adjective finite when speaking of finite simplicial spaces. By the *dimension* of a simplicial space X we mean the dimension of the simplices of the highest dimension in X and we often write it in the form of a superscript, writing X^n for an n-dimensional simplicial space.

A more geometric way of defining a simplicial space is to represent it as a subset of some Euclidean space, with the simplices being rectilinear geometric subsets of the space. Figure 3.2 shows two such examples of simplicial spaces, represented as lying in $\mathbb{R}^3$: a 2-sphere and a funny 2-dimensional simplicial space.

As the following theorem claims, any finite simplicial space X can be represented as a subset of some Euclidean space $\mathbb{R}^N$ in the sense specified above—one then says that X is *piecewise-linearly embedded* (PL-*embedded* for short) in $\mathbb{R}^N$.

Theorem 3.3. *Any finite n-dimensional simplicial space X^n can be PL-embedded in $\mathbb{R}^{2n+1}$.*

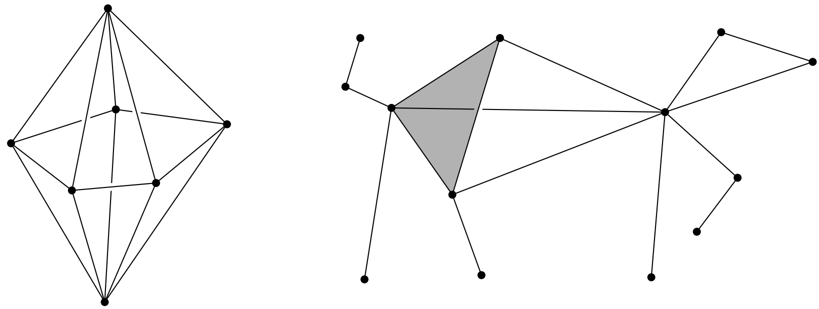


FIGURE 3.2. Two simplicial spaces as subsets of $\mathbb{R}^3$

We shall not use this theorem and therefore omit its proof. The reader may wonder where the exponent $2n+1$ comes from; there are examples of 1-dimensional simplicial spaces (e. g. the so-called $K_{3,3}$ space) that cannot be embedded in $\mathbb{R}^2$.

3.6. CW-spaces

Roughly speaking, a CW-space is a space obtained by inductively attaching k-disks ($k=0, 1, 2, \dots$) along their boundaries to the $(k-1)$-dimensional part of the previously constructed space via continuous maps of their boundaries (these maps, as well as their images, are called k-cells).

The formal definition of CW-space (also called CW-complex) is the following. Let X be a Hausdorff topological space such that

$$X = \bigcup_{i=0}^{\infty} X^i,$$

where X^0 is a discrete space and the space X^{i+1} is obtained by attaching the disjoint union of $(i+1)$-dimensional closed discs $\bigsqcup_{\alpha\in A} D_\alpha^{i+1}$ to X^i along a continuous map $\bigsqcup_{\alpha\in A} S_\alpha^i \longrightarrow X^i$, where $S_\alpha^i = \partial D_\alpha^{i+1}$. Let us call the image of D_α^{i+1} and the image of the interior of D_α^{i+1} under the natural map to $X^{i+1} \hookrightarrow X$ *closed cell* and *open cell*, respectively. The space X is called a *CW-space* (or *CW-complex*) if the two following conditions hold:

(C) any closed cell intersects a finite number of open cells;

(W) a set $C \subset X$ is closed iff any intersection of C with a closed cell is closed.

"C" is the abbreviation for "Closure Finite", "W" is the abbreviation for "Weak Topology". If the number of cells is finite, then conditions (C) and (W) hold automatically. Since we will only be considering finite cell spaces in this course, you can forget about conditions (C) and (W).

Note that any simplicial space can be considered as a CW-space (how?). Simplicial spaces are easier to visualize than CW-spaces, because simplices are simpler than cells, but CW-spaces are more economical. For example, the 77-dimensional sphere has a CW-space structure with only two cells, whereas the simplest simplicial structure of that sphere has hundreds of simplices of dimensions 0, 1, 2, . . . , 77.

3.7. Exercises

3.1. Prove that $\mathbb{D}^n/\partial\mathbb{D}^n \approx \mathbb{S}^n$.

3.2. Prove that the space $\mathbb{S}^1 \times \mathbb{S}^1$ is homeomorphic to the space obtained by the following identification of points of the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ belonging to its sides: $(x, 0) \sim (x, 1)$ and $(0, y) \sim (1, y)$. (This space is called the torus.)

3.3. Let $I = [0, 1]$. Prove that the space $\mathbb{S}^1 \times I$ is not homeomorphic to the Möbius band.

3.4. Prove that the following spaces (supplied with the natural topology) are homeomorphic:

(a) the set of lines in $\mathbb{R}^{n+1}$ passing through the origin;

(b) the set of hyperplanes in $\mathbb{R}^{n+1}$ passing through the origin;

(c) the sphere $\mathbb{S}^n$ with identified diametrically opposite points (every pair of diametrically opposite points is identified);

(d) the disc $\mathbb{D}^n$ with identified diametrically opposite points of the boundary sphere $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$.

3.5. Prove that the following spaces are homeomorphic:

(a) the set of complex lines in $\mathbb{C}^{n+1}$ passing through the origin;

(b) the sphere $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with identified points of the form λx for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$ (for any fixed point $x \in \mathbb{S}^{2n+1}$);

(c) the disc $\mathbb{D}^{2n} \subset \mathbb{C}^n$ with points of the boundary sphere $\mathbb{S}^{2n-1} = \partial\mathbb{D}^{2n}$ of the form λx for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$ identified for any fixed point $x \in \mathbb{S}^{2n-1}$.

3.6. Prove that $C(\mathbb{D}^n) \approx \mathbb{D}^{n+1}$ and $\Sigma(\mathbb{D}^n) \approx \mathbb{D}^{n+1}$. (Here and below $\approx$ denotes homeomorphisms.)

3.7. Prove that $\mathbb{R}P^1 \approx \mathbb{S}^1$ and $\mathbb{C}P^1 \approx \mathbb{S}^2$.

3.8. Prove that $C(\mathbb{S}^n) \approx \mathbb{D}^{n+1}$ and $\Sigma(\mathbb{S}^n) \approx \mathbb{S}^{n+1}$.

3.9. Is it true (for arbitrary CW-spces) that (a) $X * Y \approx Y * X$; (b) $(X * Y) * Z \approx X * (Y * Z)$; (c) $C(X * Y) \approx C(X) * Y$; (d) $\Sigma(X * Y) \approx \Sigma(X) * Y$?

3.10. Prove that $\mathbb{S}^n * \mathbb{S}^m \approx \mathbb{S}^{n+m+1}$.

3.11. Prove that $\mathbb{S}^{n+m-1} \setminus \mathbb{S}^{n-1} \approx \mathbb{R}^n \times \mathbb{S}^{m-1}$. (We suppose that the position of $\mathbb{S}^{n-1}$ in $\mathbb{S}^{n+m-1}$ is standard.)

3.12. Prove that (a) the sphere $\mathbb{S}^2$; (b) the torus $\mathbb{T}^2$; (c) the real projective space $\mathbb{R}P^n$; (d) the complex projective space $\mathbb{C}P^n$ are CW-spaces.

3.13. Find an example of a space consisting of cells that satisfies the W-axiom, and does not satisfy the C-axiom and vice versa.

Lecture $\overline{3}$

Graphs

The number of this lecture is overlined, which indicates that the lecture is optional, it should be regarded as additional reading material and a source of problems for the exercise class. In the lecture, we study a very simple class of topological spaces, called graphs. Roughly speaking, a graph G is a set of points, called vertices, some pairs of which are joined by arcs, called edges. Graphs can be defined as purely combinatorial objects, or as topological spaces. Their simplicity is due to the fact that, as combinatorial objects, they are finite and, as topological spaces, they have the smallest nontrivial dimension (one). Nevertheless, they have many surprising, beautiful, and rather intricate properties. We should also note that at the present time graph theory plays a remarkably important role in front-line research in many areas of mathematics.

$\overline{3}$.1. Main definitions

The combinatorial definition of a graph is this: a (*combinatorial*) *graph* G is pair $G=(\mathcal{V},\mathcal{E})$ consisting of a finite set $\mathcal{V}$ of undefined objects, called *vertices*, and a finite collection $\mathcal{E}$ of pairs of vertices, called *edges*; if $e=\{v,v'\}$ is an edge, we say that e *joins* v and v', or that v and v' are the *endpoints* of e; an edge $e=\{v,v'\}$ is said to be a *loop* if $v=v'$; if there are repetitions in the collection of edges $\mathcal{E}$ (i.e., there is more than one edge joining two vertices v and v'), we say that the graph G has *multiple edges.* We shall mostly be studying graphs without loops or multiple edges, and use the term “graph” in that sense; whenever a graph will be allowed to have loops or multiple edges, this will be explicitly mentioned.

Two combinatorial graphs are called *isomorphic* if there exists a bijection between the set of vertices and a bijection between the collection of edges that preserve incidence (i.e., the endpoints of any edge correspond to the endpoints of the corresponding edge).

The topological definition of a graph is this: a (*topological*) *graph* G is topological space G supplied with a finite set $\mathcal{V}$ of distinguished points, called *vertices*, and consisting of the union of a finite number of arcs, called *edges*, each arc being either a broken line joining two vertices or a closed broken line (called a *loop*) containing exactly one vertex; the arcs (including the loops) are assumed pairwise nonintersecting[1]. The graphs considered in this lecture will be subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$ supplied with the topology induced from $\mathbb{R}^2$ or $\mathbb{R}^3$. Unless stated otherwise, we will assume that they contain no loops or multiple edges.

A topological graph is said to be a *realization* of a combinatorial graph if there is a bijection between vertices and a bijection between edges preserving incidence (i.e., endpoints correspond to endpoints); in that situation, we also say that the combinatorial graph is *associated* to the topological one. It is obvious that *two graphs with isomorphic associated combinatorial graphs are homeomorphic.* The converse statement is not true (why?).

The *valency* of a vertex v of a graph G is the number of edges with endpoint v (if there are loops joining v to itself, then each loop contributes 2 to the valency). A *path* joining two vertices v and v' is a sequence of edges of the form $\{v, v_1\}, \{v_1, v_2\}, \dots, \{v_k, v'\}$; if $v = v'$ and $k > 2$, then the path is called a *cycle.* A graph G is *connected* if any two of its vertices can be joined by a path. A graph is called a *tree* if it is connected and has no cycles; the vertices of valency 1 of a tree are called *leaves.*

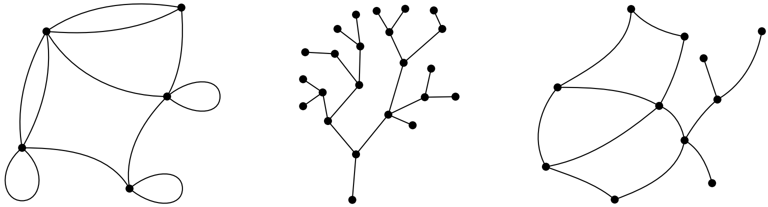
Figure $\overline{3}$.1 shows examples of (a) a graph with loops and multiple edges; (b) a tree; (c) a graph without loops or multiple edges, but containing cycles.

The three graphs appearing in the figure are subsets of the plane $\mathbb{R}^2$, but there exist graphs which cannot be placed in the plane. We shall consider them in the next section.

$\overline{3}$.2. Planar and nonplanar graphs

A topological graph is called *planar* if it lies in the plane $\mathbb{R}^2$ (and so its edges have no common internal points). A combinatorial graph G is called *planar* if it can be realized by a planar topological graph.

[1] The assumption that the arcs are polygonal (i.e., are broken lines) is purely technical, it does not restrict (up to topological equivalence) the class of graphs considered, but allows to prove certain statements about embedded graphs which are very difficult to prove without this assumption.



(a) (b) (c)

FIGURE $\overline{3}$.1. Examples of graphs

Denote by K_n the *complete graph on* n *vertices*, i.e., the graph consisting of n vertices every pair of which is joined by an edge. Denote by $K_{n,m}$ the graph consisting of $n+m$ vertices divided into two parts (n vertices in one part and m vertices in the other), the edges of $K_{n,m}$ joining each pair of vertices from different parts. The figure below represents three examples of the graphs defined above.

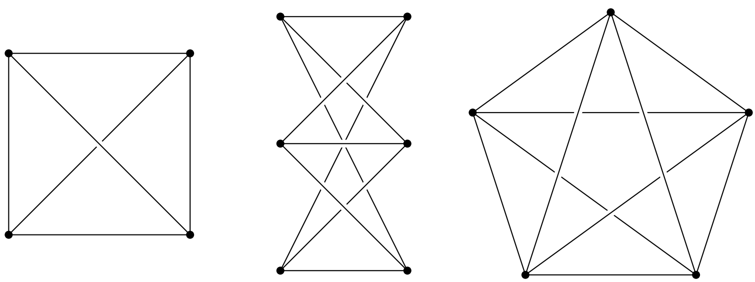


FIGURE $\overline{3}$.2. The graphs K_4, $K_{3,3}$, and K_5

The three graphs in the figure are pictured as lying in 3-space $\mathbb{R}^3$. Are they planar? The reader will easily draw a graph isomorphic to K_4 embedded in the plane—so K_4 is planar. Attempts to embed the graphs K_5 and $K_{3,3}$ will necessarily fail (the best one can do is to draw a picture of, say, $K_{3,3}$ on the plane with only one pair of edges intersecting).

Theorem $\overline{3}$.1. *The graphs* K_5 *and* $K_{3,3}$ *are not planar.*

No simple proof of this beautiful fact is known. In the sections that follow, we shall obtain two different proofs of the theorem. As usual in mathematics, in order to prove that something is impossible (in this case, it is impossible to embed K_5 or $K_{3,3}$ in $\mathbb{R}^2$), we need an invariant. We shall see in subsequent lectures that the invariant that we will use (the Euler characteristic) has many other important applications.

$\overline{3}$.3. Euler characteristic of graphs and planar graphs

If G is a graph (topological or combinatorial), we denote by V_G and E_G the number of vertices and edges of G, respectively; we omit the subscript G if the graph under consideration is clear from the context.

We define the *Euler characteristic of a graph G* by setting

$$\boxed{\chi(G) := V_G - E_G}.$$

Theorem $\overline{3}$.2. *Two connected graphs homeomorphic as topological spaces have the same Euler characteristic.*

Two such graphs differ only by the number of vertices of valency 2, but that does not affect the Euler characteristic.

Let $G \subset \mathbb{R}^2$ be a connected planar graph. Then the connected components of $\mathbb{R}^2 \setminus G$ are called *faces* of the planar graph G. Let us denote by V_G, E_G, F_G the number of vertices, edges, faces of G, respectively (we omit the subscript G if it is clear from the context). We define the *Euler characteristic of the planar graph $G \subset \mathbb{R}^2$* by setting

$$\boxed{\chi(G) := V_G - E_G + F_G}.$$

Theorem $\overline{3}$.3. *The Euler characteristic of any connected planar graph G is equal to 2:*

$$\boxed{G \subset \mathbb{R}^2 \Longrightarrow \chi(G) = 2.}$$

The proof is the object of Exercise $\overline{3}$.13 (which relies on the next theorem).

Theorem $\overline{3}$.4 (Polygonal Jordan Theorem)**.** *Let C be a closed non-self-intersecting broken line (with a finite number of segments) on $\mathbb{R}^2$. Prove that $\mathbb{R}^2 \setminus C$ consists of two connected components and the boundary of each component is C.*

$\overline{3}$.4. Exercises

$\overline{3}$.1. Is it possible to build direct roads between 53 towns so that any town is connected exactly with 3 other towns?

$\overline{3}$.2. Suppose the valencies of all the vertices of a connected graph G are even. Then there exists a path that traverses each edge of G exactly once.

$\overline{3}$.3. Prove that any connected planar graph (without loops and double edges) has a vertex of degree not greater than 5.

$\overline{3}$.4. Prove that one can color the vertices of any planar graph (without loops) using five colors so that the ends of any edge have different colors.

$\overline{3}$.5. Let K_n be the graph consisting of n vertices pairwise joined by edges. Let $K_{n,m}$ be the graph consisting of $n+m$ vertices divided into two parts (n vertices in one part and m vertices in the other), the edges of $K_{n,m}$ joining each pair of vertices from different parts.

$\overline{3}$.6. Prove that the graphs $K_{3,3}$ and K_5 are not planar.

$\overline{3}$.7. (a) Let G be a planar graph such that any face of G is bounded by an even number of edges. Prove that one can color the vertices of G using two colors so that the ends of any edge have different colors.

(b) Let γ be a smooth closed curve with transversal self-intersections. Prove that γ divides the plane into domains so that one can color those domains using two colors (two domains with a common edge must be of different colors).

$\overline{3}$.8. Let a, b, c, d be points of a closed non-self-intersecting broken line C (in the plane) ordered as indicated. Suppose that points a and c are joined by a broken line L_1, points b and d are joined by a broken line L_2 and both broken lines belong to the same connected component defined by C. Prove that L_1 and L_2 have a common point.

$\overline{3}$.9. Let G be a polygonal planar graph consisting of s connected components each of which is not an isolated vertex. Let G have v vertices and e edges. Using the polygonal Jordan theorem and induction, prove that for any embedding of G in the plane the number of faces f is equal to $f=1+s-v+e$.

$\overline{3}$.10. (a) Suppose G is a planar graph without isolated vertices, v_i is the number of its vertices of degree i, f_i is the number of faces with i edges. Prove that $\sum_i(4-i)v_i+\sum_j(4-j)f_j=4(1+s)\geqslant 8$, where s is the number of connected components of G.

(b) Prove that if all faces are quadrilaterals, then $3v_1+2v_2+v_3\geqslant 8$.

(c) Prove that if the boundary of any face is a cycle containing no less than n edges, then $e\leqslant n(v-2)/(n-2)$.

$\overline{3}$.11. Find and deduce the Euler Formula for convex polyhedra from the Euler formula for planar graphs. (The Euler Formula for convex polyhedra is a relation between numbers of vertices, edges and faces.)

$\overline{3}$.12. With the help of Exercise $\overline{3}$.10 (c), give another proof of the nonplanarity of the graphs K_5 and $K_{3,3}$.

$\overline{3}$.13. Prove Theorem $\overline{3}$.3.