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To my wonderful teacher Valerii Vasil'evich Kozlov
on the occasion of his 60*th birthday*

Khintchine's singular Diophantine systems and their applications

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Abstract. This paper is a survey of classical and recent methods in Diophantine approximation theory and its applications related to Khintchine's results on the existence of real numbers admitting extremely good approximations by rational numbers.

Bibliography: 145 titles.

Keywords: multidimensional Diophantine approximations, Khintchine's singular systems, continued fractions, best approximations, Diophantine inequalities, transference theorems, Kozlov problem, Peres–Schlag method.

Contents

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In 1926 Khintchine observed (see [1]) that phenomena which occur in two-dimensional Diophantine approximation problems differ radically from those which occur in the problem of rational approximations of an irrational number. In particular, he constructed two-dimensional real vectors which admit 'extremely good' rational approximations (in the sense of a linear form as well as in the sense of simultaneous approximations). The construction in [1] was later generalized to the case of several linear forms in several integer variables. The coefficient matrices of systems of linear forms admitting such extremely good rational approximations were called *singular matrices*.

In this paper we give a survey of certain results in Diophantine approximation theory and its applications related to Khintchine's singular matrices. This part of Diophantine approximation theory was in general constructed by Khintchine [2]–[8] and Jarník [9]–[15].

The fundamental paper [1] is one of the best (if not the very best) of Khintchine's papers in Diophantine approximation theory. Many of his papers were recently reprinted in Russian in the book [16] of his selected works in number theory. It is a pity that excellent papers by Jarník are difficult to find. For a long time many of them were forgotten.

In the beginning of this survey we discuss the definition of a singular matrix, existence theorems, and the connection with best Diophantine approximation (§ 1). In § 2 we give general results dealing with subspaces generated by best Diophantine approximations. The case of simultaneous approximations is considered separately (§ 4). In § 5 we consider Jarník's problem concerning individual and uniform Diophantine characteristics. In the same section we also give an improvement of a result of Jarník. The author's results on the degeneracy of the dimension of subspaces generated by best Diophantine approximations are discussed in the paper in several places. Problems related to inhomogeneous linear Diophantine approximations are discussed in § 6. In § 7 certain lattice theory problems connected with Diophantine approximation are considered. § 8 is devoted to classical and new results related to the transference principle. All results known to the author about the Hausdorff dimension of sets of singular matrices are gathered in § 9.

Here we should note that Diophantine approximations are important in certain problems in classical mechanics related to the problem of 'small denominators', for example, in the theory of perturbations of quasi-periodic motions (see [17]). In particular, it turned out that one can apply Khintchine's singular matrices to construct systems with 'rapid divergence of trajectories'. That was done by Kozlov and Moshchevitin in [18]. Moreover, Diophantine analysis turned out to be the main tool for the solution of the problem of oscillation of the integral of a quasi-periodic function. This problem was settled by Kozlov in 1978. We discuss it in § 11.

In the Appendix (§ 13) we discuss a series of related results. First of all, we consider the results obtained by a recent method due to Peres and Schlag [19]. Then we discuss the main notions and results in the theory of winning sets constructed by Schmidt [20].

Some of the problems considered in the present paper (as well as some other interesting topics in Diophantine approximation) are discussed in the wonderful survey by Waldschmidt [21]. There are classical books by Koksma [22], Cassels [23], and Schmidt [24] devoted to Diophantine approximation theory.

In addition to the topics mentioned above we consider also some other problems; for example, Diophantine approximations with positive integers (§ 10) and the existence of matrices with extremely singular Diophantine properties (§ 12).

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1. About the definition of a singular system

Everywhere below, $\mathbf{x} = (x_1, \dots, x_m)$ denotes an integer vector. By $\|\cdot\|$ we denote the distance to the nearest integer. We consider a matrix

$$\Theta = \begin{pmatrix} \theta_1^1 & \dots & \theta_1^m \\ \multicolumn{3}{c}{\dotfill} \\ \theta_n^1 & \dots & \theta_n^m \end{pmatrix} \tag{1}$$

with real numbers θ_j^i ($1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant m$), and the corresponding system of linear forms

$$\mathbf{L}(\mathbf{x}) = \mathbf{L}_\Theta(\mathbf{x}) = \{L_j(\mathbf{x}),\ 1 \leqslant j \leqslant n\}, \qquad L_j(\mathbf{x}) = \sum_{i=1}^{m} \theta_j^i x_i. \tag{2}$$

Denote by ${}^t\Theta$ the matrix transpose of Θ.

From the Minkowski convex body theorem it follows that for any matrix Θ and any real $t \geqslant 1$ the system of Diophantine inequalities

$$\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\| \leqslant \frac{1}{t}, \qquad 0 < \max_{1\leqslant i\leqslant m} |x_i| \leqslant t^{n/m},$$

has an integer solution $\mathbf{x} \in \mathbb{Z}^m$.

1.1. Khintchine's definition. First of all we formulate the original definition of a singular system as given by Khintchine in [7]. A matrix Θ (or in the original terminology of Khintchine a set of real numbers θ_j^i with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$) is called a *singular system* if for any $\varepsilon > 0$ there exists a $t_0 = t_0(\varepsilon)$ such that for $t \geqslant t_0$ the system of Diophantine inequalities

$$\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\| \leqslant \frac{1}{t}, \qquad 0 < \max_{1\leqslant i\leqslant m} |x_i| < \varepsilon t^{n/m},$$

has an integer solution $\mathbf{x} \in \mathbb{Z}^m$.

Here we note that in Khintchine's terminology a matrix Θ which is not singular is said to be *regular* (he used the term *regular system* of real numbers θ_j^i).

Therefore, a matrix Θ is regular if there exists a $\mu(\Theta) > 0$ such that for some sequence of positive numbers t_ν tending to infinity there are no non-zero integer points in $\mathbb{R}^n$ satisfying the inequalities

$$\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\| \leqslant \frac{1}{t_\nu}, \qquad 0 < \max_{1\leqslant i\leqslant m} |x_i| \leqslant \mu(\Theta) t_\nu^{n/m}.$$

This can be said in other words. A matrix Θ is regular if there exists a $\mu(\Theta) > 0$ such that for some sequence of positive numbers t_ν tending to infinity there are no non-zero integer points in $\mathbb{R}^n$ satisfying the inequalities

$$\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\| \leqslant \frac{\mu(\Theta)}{t_\nu}, \qquad 0 < \max_{1\leqslant i\leqslant m} |x_i| \leqslant \mu(\Theta) t_\nu^{n/m}. \tag{3}$$

This is the way the definition of a regular system was formulated by Davenport and Schmidt [25].

In the present paper we mainly use matrix terminology. Nevertheless, it will sometimes be convenient to use the notion of a singular system of real numbers.

1.2. Existence theorems. It is easy to see that in the case $n = m = 1$ there exist no irrational numbers θ_1^1 which form a singular system (in the sense of Khintchine's definition). We make some additional comments on this in § 3.1. Khintchine (see [1]) was the first to prove the existence of singular matrices in the cases $m = 2$, $n = 1$ and $m = 1$, $n = 2$. Here we state the original results ([1], Hilfssatz I, Satz 2; see also [23], Chap. V, Theorem XIV).

Theorem 1. *Suppose that $\psi(t)$ is a continuous function of a real argument t and is decreasing to zero as $t \to +\infty$. Then there exist two real numbers θ^1 and θ^2 which are linearly independent together with 1 over $\mathbb{Z}$ and such that for sufficiently large t there exists an integer solution $(x_1, x_2) \in \mathbb{Z}^2$ of the Diophantine system*

$$\|x_1\theta^1 + x_2\theta^2\| < \psi(t), \qquad 0 < \max_{j=1,2} |x_j| < t.$$

Theorem 2. *Let $\psi(t)$ be a continuous function of a real argument t and suppose that it is decreasing to zero as $t \to +\infty$. Suppose that the function $t\psi(t)$ increases monotonically to infinity as $t \to +\infty$. Then there exist two real numbers θ_1 and θ_2 which are linearly independent together with 1 over $\mathbb{Z}$ and such that for sufficiently large t there exists an integer solution x of the Diophantine system*

$$\max_{j=1,2} \|x\theta_j\| < \psi(t), \qquad 0 < x < t.$$

Thus, in the case $m = 2$, $n = 1$ Theorem 1 ensures the existence of singular systems for a given function $\psi(t)$ with an arbitrary rate of convergence to zero. In the case $m = 1$, $n = 2$ Theorem 2 states the existence of singular systems if the corresponding function $t \mapsto t\psi(t)$ increases monotonically to infinity.

For this reason the following definition is convenient. Suppose that a function $\psi(t)$ decreases to zero continuously and that $\psi(t) = o(t^{-m/n})$ as $t \to +\infty$. We define a matrix Θ (or a system of mn real numbers) to be *ψ-singular* if for sufficiently large t the Diophantine system

$$\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\| \leqslant \psi(t), \qquad 0 < \max_{1\leqslant i\leqslant m} |x_i| < t,$$

has an integer solution $\mathbf{x} \in \mathbb{Z}^m$.

Theorem 1 above admits the following direct multidimensional generalization, in a stronger form.

Theorem 3. *Let n be a positive integer and m an integer $\geqslant 2$. Suppose that the continuous function $\psi(t)$ of a real argument t decreases monotonically to zero as $t \to +\infty$. Consider the set $\mathscr{M} \subset \mathbb{R}^{mn}$ of all matrices Θ such that*

- *the numbers θ_j^i with $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$ are linearly independent together with 1 over $\mathbb{Z}$,*
- *the matrix Θ is ψ-singular.*

Then for any open set $\mathscr{G} \subset \mathbb{R}^{mn}$ the intersection $\mathscr{M} \cap \mathscr{G}$ has the cardinality of the continuum.

Here we note that the situation in the case $m = 1$ is different. Suppose that $m = 1$ and let $\Theta = \{\theta_1, \dots, \theta_n\}$ be some collection of real numbers. We denote by $\dim_{\mathbb{Z}} \Theta$ the maximal number of elements among $\theta_1, \dots, \theta_n, 1$ that are linearly independent over $\mathbb{Z}$. We can now state a generalization of Theorem 2.

Theorem 4. *Let $m = 1$ and $n \geqslant 2$.*

(i) *Let $\psi(t)$ be a positive continuous function of a real variable t and assume that it decreases to zero as $t \to +\infty$. Suppose that*

$$\lim_{t\to\infty} \psi(t)t = +\infty. \tag{4}$$

Consider the set $\mathscr{M} \subset \mathbb{R}^n$ *of collections* $\Theta = \{\theta_j,\ 1 \leqslant j \leqslant n\}$ *of real numbers such that*

- $\dim_{\mathbb{Z}} \Theta = n + 1$,
- *the system* Θ *is* ψ*-singular.*

Then for any open set $\mathscr{G} \subset \mathbb{R}^n$ *the intersection* $\mathscr{M} \cap \mathscr{G}$ *has the cardinality of the continuum.*

(ii) *Let* $n \geqslant 2$ *and suppose that the positive function* $\psi(t)$ *satisfies the condition*

$$\limsup_{t\to+\infty} \psi(t)t < +\infty. \tag{5}$$

Then $\dim_{\mathbb{Z}} \Theta \leqslant 2$ *if the system* $\Theta = \{\theta_1, \dots, \theta_n\}$ *is* ψ*-singular. Moreover, if*

$$\limsup_{t\to+\infty} \psi(t)t = 0, \tag{6}$$

then $\dim_{\mathbb{Z}} \Theta = 1$, *that is, all the numbers* θ_j *with* $1 \leqslant j \leqslant n$ *are rational.*

One can find Theorem 2 and part (i) of Theorem 4 in Jarník's paper [14]. A particular case was considered by Chabauty and Lutz [26]. We note that Jarník proves a somewhat stronger result. Under the conditions of Theorem 3 he proves the existence of matrices Θ consisting of *algebraically independent* real numbers θ^i_j.

The statement (ii) in Theorem 4 obviously follows from Corollary 5 in § 4.1 below (p. 458), which was also proved by Jarník (see [9], [14]). Under the condition (6) the result follows immediately from Corollary 1 in § 4.1. Thus, we see that in the case $m = 1$, $n \geqslant 2$ non-trivial singular systems do not exist under the condition (5).

Lesca [27] obtained a somewhat more precise result than Theorem 4.

Theorem 5 (J.Lesca [27]). *Let* $m = 1$ *and* $n \geqslant 2$, *and let* $\psi(t)$ *be a positive continuous function of* t *decreasing to zero as* $t \to +\infty$. *Suppose that*

$$\limsup_{t\to\infty} \psi(t)t = +\infty.$$

Then the set of all ψ*-singular systems* $\Theta = \{\theta_j,\ 1 \leqslant j \leqslant n\}$ *consisting of algebraically independent real numbers, intersected with any open set* $\mathscr{G} \subset \mathbb{R}^n$, *has the cardinality of the continuum.*

Theorem 5 should be compared with Jarník's formula (53) and with Theorem 17. Here we note that Lesca [28] has a p-adic version of Theorem 3.

We cite an existence theorem proved by Apfelbeck.

Theorem 6 (Apfelbeck [29]). *Let* $n, m \geqslant 2$. *Suppose that* $\psi(t)$ *is a positive continuous function* $\psi(t)$ *such that the function* $t \mapsto t\psi(t)$ *is decreasing. Then there exists a* ψ*-singular matrix* Θ *which consists of numbers* θ^i_j *linearly independent together with* 1 *and such that the matrix* ${}^t\Theta$ *is also* ψ*-singular.*

We comment on the use of the word 'singular' in the definition of singular systems. The explanation is that for any dimensions m and n the set of all singular matrices is a set of zero Lebesgue measure, as was proved by Khintchine himself (see [7]). This result follows easily from the Borel–Cantelli lemma. One can find a complete proof in [23] (Chap. V, § 7). A stronger result was obtained by Davenport and Schmidt [25]. We shall discuss this result in § 7.1.

1.3. Singular systems and best approximations. For an integer vector $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ we consider the quantities

$$M(\mathbf{x}) = \max_{1\leqslant i\leqslant m} |x_i|, \qquad \zeta(\mathbf{x}) = \max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\|.$$

An integer vector $\mathbf{x} = (x_1, \dots, x_m)$ is called a *best approximation* for the matrix Θ if

$$\zeta(\mathbf{x}) = \min_{\mathbf{x}'} \zeta(\mathbf{x}'), \tag{7}$$

where the minimum is taken over all non-zero integer points $\mathbf{x}' = (x'_1, \dots, x'_m) \in \mathbb{Z}^m$ such that

$$0 < M(\mathbf{x}') \leqslant M(\mathbf{x}). \tag{8}$$

For a best approximation $\mathbf{x}$ satisfying this definition the point $-\mathbf{x}$ will also be a best approximation. However, for the pair of best approximations $\pm\mathbf{x}$ the values $M(\mathbf{x})$ and $\zeta(\mathbf{x})$ are the same and do not depend on the sign $\pm$.

We note that in general it may happen that for two integer points $\mathbf{x}_1 \neq \pm\mathbf{x}_2$ with the same value $M(\mathbf{x}_1) = M(\mathbf{x}_2) \neq 0$ one has

$$\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x}_1)\| = \max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x}_2)\|. \tag{9}$$

But in the case when all the numbers θ_j^i $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ are linearly independent together with 1 over $\mathbb{Z}$, the equality (9) is not possible. Hence, in the case when all the numbers θ_j^i $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ are linearly independent together with 1 over $\mathbb{Z}$, all the best approximations can be arranged in an infinite sequence

$$\pm\mathbf{x}_1, \pm\mathbf{x}_2, \dots, \pm\mathbf{x}_\nu, \pm\mathbf{x}_{\nu+1}, \dots$$

in such a way that

$$M(\mathbf{x}_1) < M(\mathbf{x}_2) < \cdots < M(\mathbf{x}_\nu) < M(\mathbf{x}_{\nu+1}) < \cdots, \tag{10}$$

$$\zeta(\mathbf{x}_1) > \zeta(\mathbf{x}_2) > \cdots > \zeta(\mathbf{x}_\nu) > \zeta(\mathbf{x}_{\nu+1}) > \cdots. \tag{11}$$

For brevity we use the notation

$$M_\nu = M(\mathbf{x}_\nu), \qquad \zeta_\nu = \zeta(\mathbf{x}_\nu).$$

In some places below we need to work with best approximations for matrices Θ which do not satisfy the linear independence condition. In such a case it may not be possible to determine the sequence of points $\mathbf{x}_\nu$ uniquely. Nevertheless, the sequences of values $M(\mathbf{x}_\nu)$, $\zeta(\mathbf{x}_\nu)$ (see (10), (11)) are uniquely determined. (But it may happen that there are several different pairs of integer points $\pm\mathbf{x}_\nu$ with the same values of $M(\mathbf{x}_\nu)$ and $\zeta(\mathbf{x}_\nu)$; also, the sequences $M(\mathbf{x}_\nu)$ and $\zeta(\mathbf{x}_\nu)$ may be finite in general.)

The discussion above leads to the following definition. We define a matrix Θ to be *good* if the sequences (10), (11) are infinite and for sufficiently large ν the vectors $\mathbf{x}_\nu$ are uniquely determined up to the sign.

We should make one more remark. Let $\mathbf{e}^k$ $(1 \leqslant k \leqslant n)$ be unit vectors with the kth coordinate of $\mathbf{e}^k$ equal to 1 and all other coordinates 0. We consider the collection

$$\theta^1, \dots, \theta^m, \mathbf{e}^1, \dots, \mathbf{e}^n \tag{12}$$

of $m+n$ vectors in $\mathbb{R}^n$, where θ^j denotes the jth column of the matrix Θ. It is easy to see that the sequences of best approximations (10), (11) are infinite if and only if the collection (12) consists of vectors linearly independent over $\mathbb{Z}$. In this situation Jarník defines the matrix Θ to be *non-degenerate*. Here we use this definition also. If the matrix Θ is not non-degenerate, then the sequences (10) and (11) are finite and for the last value of ν we have $\zeta_\nu = 0$.

In the case when m or n is equal to 1 the matrix Θ becomes a single column or a single row and can be identified with a tuple of m or n real numbers, and we can regard $\dim_{\mathbb{Z}} \Theta$ as a characteristic of this tuple of real numbers. This is also the sense of the notation $\dim_{\mathbb{Z}} \Theta$ in Theorem 4. We shall use this notation below in the same sense. In particular, we see that $\dim_{\mathbb{Z}} \Theta$ is defined only in the cases $m = 1$ and $n = 1$. For example, in the case $m = 1$ the non-degeneracy of Θ means that among the numbers $\theta_j = \theta_j^1$ there exists at least one irrational number, that is, $\dim_{\mathbb{Z}} \Theta \geqslant 2$. For $n = 1$ the non-degeneracy of Θ means that the numbers $1, \theta_1^1, \dots, \theta_1^m$ are linearly independent over $\mathbb{Z}$, that is, $\dim_{\mathbb{Z}} \Theta = m + 1$.

We need one more characteristic of linear independence. Given a matrix Θ, we define $\mathrm{DIM}_{\mathbb{Z}} \Theta$ to be the maximal number of vectors linearly independent over $\mathbb{Z}$ in the collection (12). Note that for $n = 1$

$$\mathrm{DIM}_{\mathbb{Z}} \Theta = \dim_{\mathbb{Z}} \Theta, \tag{13}$$

and for $m = 1$ the dimension $\mathrm{DIM}_{\mathbb{Z}} \Theta$ can take only the two values n and $n+1$. In this case instead of the equality (13) we clearly have

$$\mathrm{DIM}_{\mathbb{Z}} {}^t\Theta = \dim_{\mathbb{Z}} {}^t\Theta. \tag{14}$$

Obviously, for any matrix Θ

$$n \leqslant \mathrm{DIM}_{\mathbb{Z}} \Theta \leqslant n + m. \tag{15}$$

A matrix Θ is non-degenerate if and only if

$$\mathrm{DIM}_{\mathbb{Z}} \Theta = m + n.$$

We make a comment concerning the difference between the notions of a good matrix and a non-degenerate matrix. For $m = 1$ or $n = 1$ non-degeneracy of a matrix forces the matrix to be good. In other cases this is not so. Here is an example.

Let ξ be an irrational number, let $m = n = 2$, and let Θ be

$$\Theta = \begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \theta_2^1 & \theta_2^2 \end{pmatrix} = \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix}.$$

Then

$$\mathrm{DIM}_{\mathbb{Z}} \Theta = \mathrm{DIM}_{\mathbb{Z}} {}^t\Theta = 4,$$

and the matrix is non-degenerate. But it is easy to see that for each value of ν the corresponding vectors

$$(\pm q_\nu, \pm q_\nu), \quad (\pm q_\nu, 0), \quad (0, \pm q_\nu),$$

where q_ν is the denominator of a (continued fraction) convergent to ξ, have the same values of M_ν and ζ_ν. Of course, in this example the collection 1, θ^i_j with $1 \leqslant i, j \leqslant 2$ consists of numbers linearly dependent over $\mathbb{Z}$, since some θ^i_j are equal to zero.

Let us continue our study of best approximations. By $\mathbf{y}_\nu = (y_{1,\nu}, \dots, y_{n,\nu}) \in \mathbb{Z}^n$ we denote an integer vector with coordinates $y_{j,\nu}$ such that

$$\|L_j(\mathbf{x}_\nu)\| = |L_j(\mathbf{x}_\nu) + y_{j,\nu}|.$$

We use the notation

$$\mathbf{z}_\nu = (x_{1,\nu}, \dots, x_{m,\nu}, y_{1,\nu}, \dots, y_{n,\nu}) \in \mathbb{Z}^d, \qquad d = m + n,$$

for the 'extended' best approximation vector.

A vector

$$\mathbf{z} = (x_1, \dots, x_m, y_1, \dots, y_n)$$

is an 'extended' best approximation vector if there are no integer points inside the parallelepiped

$$\Big\{\mathbf{z}' = (x'_1, \dots, x'_m, y'_1, \dots, y'_n) : \ M(\mathbf{x}') \leqslant M(\mathbf{x}), \\ \max_{1\leqslant j\leqslant n} |L_j(\mathbf{x}') + y'_j| \leqslant \max_{1\leqslant j\leqslant n} |L_j(\mathbf{x}) + y_j|\Big\} \tag{16}$$

besides the point $\mathbf{0}$. Moreover, there are no integer points inside the parallelepiped

$$\Big\{\mathbf{z}' = (x'_1, \dots, x'_m, y'_1, \dots, y'_n) : \ M(\mathbf{x}') \leqslant M(\mathbf{x}_{\nu+1}), \\ \max_{1\leqslant j\leqslant n} |L_j(\mathbf{x}) + y'_j| \leqslant \zeta_\nu = \max_{1\leqslant j\leqslant n} |L_j(\mathbf{x}_\nu) + y_{j,\nu}|\Big\} \tag{17}$$

besides $\mathbf{0}$. From this we deduce by Minkowski's convex body theorem that

$$\zeta_\nu^n M_{\nu+1}^m \leqslant 1. \tag{18}$$

Moreover, we note two simple properties.

1. Each extended best approximation vector $\mathbf{z}_\nu$ is a *primitive* vector, that is,

$$\text{g.c.d.}(x_{1,\nu}, \dots, x_{m,\nu}, y_{1,\nu}, \dots, y_{n,\nu}) = 1.$$

2. Every pair of two consecutive best approximation vectors $\mathbf{z}_\nu$, $\mathbf{z}_{\nu+1}$ can be extended to a basis of the integer lattice $\mathbb{Z}^d$.

The property of singularity for the matrix Θ can easily be reformulated in terms of best approximations.

Proposition 1. *Suppose that a continuous and monotone function $\psi(t)$ satisfies the condition $\psi(t) = o(t^{-m/n})$ for $t \to +\infty$. A non-degenerate matrix Θ is ψ-singular if and only if for sufficiently large ν*

$$\zeta_\nu \leqslant \psi(M_{\nu+1}). \tag{19}$$

The connection between the singularity of Θ and best approximations does not appear in Khintchine's papers in an explicit way. This connection was implicitly used in Jarník's papers and in Cassels' book [23]. In particular, Jarník (see [9], [11], [13], [14]) uses a piecewise constant function

$$\psi_\Theta(t) = \min_{\mathbf{x}\in\mathbb{Z}^m:\,0<M(\mathbf{x})\leqslant t}\ \max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\|, \tag{20}$$

and in [23] the similar function

$$\eta(\rho) = \min_{\mathbf{x}\in\mathbb{Z}^m:\,0<|x_1|^2+\dots+|x_m|^2\leqslant\rho^2}\ \max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\|$$

appears in several proofs (Chap. V, §§ 6, 7).

The jumps of these functions in fact determine the best approximations (in Jarník's papers in the sup-norm as in the present paper, but in the book [23] in the Euclidean norm).

We note that a matrix Θ is non-degenerate if and only if the value $\psi_\Theta(t)$ is never zero for $t \geqslant 1$. A non-degenerate matrix Θ is ψ-singular if and only if for sufficiently large t Jarník's function (20) satisfies

$$\psi_\Theta(t) \leqslant \psi(t).$$

2. Subspaces generated by best approximations

In this section we discuss properties of linear subspaces of $\mathbb{R}^d$ where the best approximation vectors $\mathbf{z}_\nu$ lie.

2.1. Bounds for dimensions. By the dimension $\dim\Lambda$ of a lattice Λ we mean the dimension of the minimal linear subspace $\operatorname{span}\Lambda \subset \mathbb{R}^d$ containing Λ. Consider a subspace π of $\mathbb{R}^d$. The intersection $\pi \cap \mathbb{Z}^d$ is a lattice $\Lambda = \Lambda(\pi)$ (possibly containing just the zero point $\mathbf{0}$). We define a subspace $\pi \subseteq \mathbb{R}^d$ to be *completely rational* if

$$\dim\pi = \dim\Lambda(\pi).$$

For a given linear subspace π we define $\mathfrak{H}(\pi)$ as the maximal completely rational subspace which *is contained in* π, and we define $\mathfrak{R}(\pi)$ as the minimal completely rational linear subspace which *contains* π. So

$$\mathfrak{H}(\pi) \subseteq \pi \subseteq \mathfrak{R}(\pi).$$

We now define vectors

$$\bar\theta_i = (\theta_i^1,\dots,\theta_i^m,0,\dots,0,1,0,\dots,0), \qquad 1 \leqslant i \leqslant n$$

(here 1 is at the $(m+i)$th place).

Consider in the space $\mathbb{R}^d$ with $d = m+n$ the linear subspace $\mathscr{N}(\Theta)$ generated by the vectors $\bar{\theta}_1, \dots, \bar{\theta}_n$, and its orthogonal complement $\mathscr{L}(\Theta)$. Obviously,

$$\dim \mathscr{N}(\Theta) = n, \qquad \dim \mathscr{L}(\Theta) = m.$$

We now consider the subspaces

$$\mathfrak{H}_\Theta = \mathfrak{H}(\mathscr{L}(\Theta)), \qquad \mathfrak{R}_\Theta = \mathfrak{R}(\mathscr{L}(\Theta)).$$

For them,

$$\mathrm{DIM}_{\mathbb{Z}}\,\Theta + \dim \mathfrak{H}_\Theta = d, \qquad \mathrm{DIM}_{\mathbb{Z}}\,{}^t\Theta = \dim \mathfrak{R}_\Theta. \tag{21}$$

In particular, from the first equality in (21) we see that a matrix Θ is non-degenerate if and only if $\mathfrak{H}_\Theta = \{\mathbf{0}\}$.

The following proposition is well known. In fact, it was proved by Jarník in [13]. The case $m = 1$ can be found in Lagarias' paper [30]. The case $m = 2$, $n = 1$ was treated by Davenport and Schmidt [31] (see also the author's papers [32], [33]).

For a good matrix Θ we define

$$R(\Theta) = \min\{r : \text{there exists a lattice } \Lambda \subseteq \mathbb{Z}^{n+m},\ \dim \Lambda = r,\ \text{and } \nu_0 \in \mathbb{N} \\ \text{such that } \mathbf{z}_\nu \in \Lambda \text{ for all } \nu \geqslant \nu_0\}.$$

Let Λ_Θ be the lattice in the definition of $R(\Theta)$ and put $\pi_\Theta = \operatorname{span} \Lambda_\Theta$. Let

$$K(\Theta) = \dim\bigl(\pi_\Theta \cap \mathscr{L}(\Theta)\bigr) \geqslant 1. \tag{22}$$

The last inequality follows from the fact that there are best approximation vectors in the subspace π_Θ that are arbitrarily close to the subspace $\mathscr{L}(\Theta)$.

Theorem 7. *For a good matrix Θ the following statements are valid*:

(i) $2 \leqslant R(\Theta) \leqslant \mathrm{DIM}_{\mathbb{Z}}\,{}^t\Theta$;
(ii) *if* $m = 1$, *then* $R(\Theta) = \mathrm{DIM}_{\mathbb{Z}}\,{}^t\Theta = \dim_{\mathbb{Z}} \Theta$;
(iii) *if* $m > n$, *then* $R(\Theta) \geqslant 3$;
(iv) *if* $K(\Theta) = 1$ *and* $R(\Theta) > 2$, *then* $m < n$.

We make some remarks about the proofs of the statements in Theorem 7.

1. The lower bound in (i) follows from the linear independence of the vectors $\mathbf{z}_\nu$, $\mathbf{z}_{\nu+1}$.

To prove the upper bound in (i) and the statement (ii), we introduce some notation which will be useful not only in this proof but also below.

The distance between sets $\mathscr{A}, \mathscr{B} \subset \mathbb{R}^d$ is denoted by $\operatorname{dist}(\mathscr{A}, \mathscr{B})$. (It is convenient to take the distance in the sup-norm.) For a lattice $\Lambda \subset \mathbb{R}^d$ such that $\mathbb{Z}^d \cap \operatorname{span} \Lambda = \Lambda$ and $\dim \Lambda < d$, the distance between $\mathbb{Z}^d \setminus \Lambda$ and the subspace $\operatorname{span} \Lambda$ is greater than zero. Denote it by

$$\rho(\Lambda) = \operatorname{dist}(\mathbb{Z}^d \setminus \Lambda, \operatorname{span} \Lambda) > 0. \tag{23}$$

For a completely rational linear subspace $\pi \subset \mathbb{R}^d$ we let

$$\rho(\pi) = \rho(\pi \cap \mathbb{Z}^d). \tag{24}$$

2. We now comment about the proof of the upper bound in the statement (i).

Since $\rho(\mathfrak{R}_\Theta) > 0$ but $\operatorname{dist}(\mathbf{z}_\nu, \mathfrak{R}_\Theta) \leqslant \operatorname{dist}\bigl(\mathbf{z}_\nu, \mathscr{L}(\theta)\bigr) \to 0$ as $\nu \to +\infty$, we see that $\mathbf{z}_\nu \in \mathfrak{R}_\Theta \cap \mathbb{Z}^{n+m}$ for large enough ν, and the proof is complete.

3. Here we comment about the proof of the statement (ii). Note that if $m = 1$, then the subspace $\mathscr{L}(\Theta)$ has dimension 1, and $d = n + 1$.

Suppose that $\dim_{\mathbb{Z}} \Theta = r$. Then the vector Θ belongs to a certain completely rational linear subspace $\pi \subset \mathbb{R}^{n+1}$ of dimension r (in fact this subspace is $\mathfrak{R}_\Theta$), but does not belong to any completely rational subspace of a lower dimension.

The bound $R(\Theta) \leqslant r$ follows from the upper bound in (i).

Suppose that $R(\Theta) < r$. But $\operatorname{dist}\bigl(\mathbf{z}_\nu, \mathscr{L}(\theta)\bigr) \to 0$ as $\nu \to +\infty$, so it follows that Θ belongs to a completely rational subspace of dimension $< r$, which is not possible.

Hence $R(\Theta) = r$, and (ii) is proved.

4. To prove (iii) one should take into account the inequality (18), which in the case $m > n$ implies that

$$\zeta_\nu M_{\nu+1} \to 0, \qquad \nu \to +\infty. \tag{25}$$

In the case $R(\Theta) = 2$ we see that the two-dimensional subspace $\pi_\Theta = \operatorname{span} \Lambda_\Theta$ (here Λ_Θ is the lattice in the definition of $R(\Theta)$) contains the one-dimensional subspace $\pi_\Theta \cap \mathscr{L}(\Theta)$, which does not contain non-zero points of the lattice Λ_Θ (since $\mathfrak{H}_\Theta = \{\mathbf{0}\}$). Now (25) contradicts Proposition 2 in § 3.3 (the norm $|\cdot|_\bullet$ is induced by the sup-norm in $\mathbb{R}^d$).

5. To prove the inequality (iv) we again use the inequality (18), which for $m \geqslant n$ implies that

$$\zeta_\nu M_{\nu+1} \leqslant 1 \quad \forall\, \nu.$$

Then we consider the completely rational subspace π_Θ and the one-dimensional subspace $\ell_\Theta = \pi_\Theta \cap \mathscr{L}(\Theta)$. The latter does not belong to any proper completely rational subspace of π_Θ. We use Proposition 3 in § 4.1 (p. 459) to see that

$$\zeta_\nu M_{\nu+1} \to +\infty, \qquad \nu \to +\infty.$$

This is a contradiction.

We note that in item 4 above we have proved a slightly more general statement: in addition to the statement (iii) of Theorem 7 we get the following result.

Theorem 8. *Let $2 \leqslant m \leqslant n$, and let Θ be a good matrix that is ψ-singular, with some function such that $\psi(t) = o(t^{-1})$ as $t \to +\infty$. Then $R(\Theta) \geqslant 3$.*

Under the conditions of Theorem 8, we suppose in the case $m = n \geqslant 2$ that the matrix Θ is singular in the sense of Khintchine's original definition. In the case $n > m$ we require something more.

Remark. In any case when $R(\Theta) \geqslant 3$, there exist infinitely many values of ν such that the three consecutive best approximation vectors $\mathbf{z}_{\nu-1}$, $\mathbf{z}_\nu$, $\mathbf{z}_{\nu+1}$ are linearly independent.

As will be shown in the next subsection, the subspace π_Θ may really have small dimension $R(\Theta)$. Nevertheless, such degeneracy of the dimension can occur only under strong restrictions on the elements of the matrix Θ.

Theorem 9. *Let Θ be a good matrix, and let $R(\Theta) \leqslant n + K(\Theta) - 1$. Then the matrix Θ consists of elements θ_j^i satisfying some algebraic relation of degree* $\leqslant \min(m, R(\Theta) - K(\Theta) + 1)$.

Theorem 9 is almost obvious. The subspace $\mathscr{L}(\Theta)$ is generated by the vectors

$$\underline{\theta}^1 = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \theta_1^1 \\ \vdots \\ \theta_n^1 \end{pmatrix}, \quad \underline{\theta}^2 = \begin{pmatrix} 0 \\ -1 \\ \vdots \\ 0 \\ \theta_1^2 \\ \vdots \\ \theta_n^2 \end{pmatrix}, \quad \dots, \quad \underline{\theta}^m = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ -1 \\ \theta_1^m \\ \vdots \\ \theta_n^m \end{pmatrix}.$$

One can take a basis in π_Θ consisting of $R(\Theta)$ integer vectors of the form

$$\mathbf{u}^j = \begin{pmatrix} u_1^j \\ \vdots \\ u_{m+n}^j \end{pmatrix}, \qquad 1 \leqslant j \leqslant R(\Theta).$$

Since $\dim \operatorname{span}\bigl(\pi_\Theta \cup \mathscr{L}(\Theta)\bigr) = R(\Theta) + m - K(\Theta)$, we see from (22) that the vectors $\underline{\theta}^1, \dots, \underline{\theta}^m, \mathbf{u}^1, \dots, \mathbf{u}^{R(\Theta)}$ are linearly dependent over $\mathbb{R}$. Moreover, the subcollection

$$\underline{\theta}^1, \dots, \underline{\theta}^m, \mathbf{u}^1, \dots, \mathbf{u}^r, \qquad r = R(\Theta) - K(\Theta) + 1, \quad 1 \leqslant r \leqslant n, \tag{26}$$

also consists of vectors linearly dependent over $\mathbb{R}$. From the conditions of the theorem it follows that $m + R(\Theta) - K(\Theta) + 1 \leqslant m + n$. Consider the matrix of size $(m+n) \times \bigl(m + R(\Theta) - K(\Theta) + 1\bigr)$ consisting of the coordinates of the vectors (26). All the maximal-order minors of this matrix are equal to zero.

Since the vectors $\mathbf{u}^1, \dots, \mathbf{u}^r$ are linearly independent, there is a collection of indices

$$1 \leqslant i_1 < \dots < i_r \leqslant d$$

such that

$$\det \begin{pmatrix} u_{i_1}^1 & \dots & u_{i_1}^r \\ \dots & \dots & \dots \\ u_{i_r}^1 & \dots & u_{i_r}^r \end{pmatrix} \neq 0.$$

The collection of r indices $1 \leqslant i_1 < \dots < i_r \leqslant d$ can be extended to a collection of $r + m$ indices of the form

$$1, 2, \dots, m, j_1, \dots, j_r, \qquad m < j_1 < \dots < j_r \leqslant d.$$

Thus,

$$\det \begin{pmatrix} -1 & \dots & 0 & u_1^1 & \dots & u_1^r \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & u_m^1 & \dots & u_m^r \\ \theta_{j_1}^1 & \dots & \theta_{j_1}^m & u_{j_1}^1 & \dots & u_{j_1}^r \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \theta_{j_r}^1 & \dots & \theta_{j_r}^m & u_{j_r}^1 & \dots & u_{j_r}^r \end{pmatrix} = 0.$$

From the columns u^j of the determinant we can extract a non-zero minor which is different from the bottom right-hand minor. Therefore, the condition that this determinant is equal to zero gives a non-trivial algebraic relation of degree $\leqslant \min(m, r)$ on the elements of the matrix Θ.

From the statement (i) in Theorem 7 and from the second equality in (21) we immediately deduce the following corollary.

Corollary 1. *Let Θ be a good matrix. If $K(\Theta) = m$, then $\pi_\Theta = \mathfrak{R}_\Theta$ and $R(\Theta) = \mathrm{DIM}_{\mathbb{Z}}\, {}^t\Theta$.*

From Theorems 7 and 9 and the inequality (22) we obtain the following.

Corollary 2. *Let Θ be a good matrix. Then:*
(i) *if the elements of Θ are algebraically independent, then $R(\Theta) \geqslant n+1$;*
(ii) *if $(m,n) \neq (1,1)$ and the elements of Θ are algebraically independent, then $R(\Theta) \geqslant 3$.*

For the case $m = n$ we deduce the following assertions from Theorem 7 (statement (iv)) and Theorem 9.

Corollary 3. (i) *Let Θ be a good matrix. If $m = n > 1$ and $R(\Theta) \leqslant n+1$, then the elements of Θ are algebraically dependent.*
(ii) *Let Θ be a good matrix. If $m = n > 1$ and $K(\Theta) = 1$, then $R(\Theta) = 2$ and the elements of Θ are algebraically dependent.*

Corollary 4. *Let $m = n = 2$. Let Θ be a good matrix. Then:*
(i) *if $K(\Theta) = 1$, then $R(\Theta) = 2$;*
(ii) *if $K(\Theta) = 2$, then $\pi_\Theta \supseteq \mathscr{L}(\Theta)$ and $R(\Theta) = \mathrm{DIM}_{\mathbb{Z}}\, {}^t\Theta$.*
In particular, if $\mathrm{DIM}_{\mathbb{Z}}\, {}^t\Theta = 4$, then either $R(\Theta) = 2$ or $R(\Theta) = 4$.

2.2. Degeneracy of dimension, $R(\Theta) = 2$. Let $\xi \in (0,1)$ be an irrational number and let p_ν/q_ν with $\nu = 1,2,3,\dots$ be the convergents to ξ. Let $m \geqslant 2$ and $n \geqslant 3$, and suppose that the elements θ_j^i with $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant n$ satisfy the relations

$$\theta_j^2 = -\xi\theta_j^1, \qquad 1 \leqslant j \leqslant n. \tag{27}$$

We consider the matrix

$$\Theta = \begin{pmatrix} \theta_1^1 & \theta_1^2 & \theta_1^3 & \dots & \theta_1^m \\ \dots & \dots & \dots & \dots & \dots \\ \theta_n^1 & \theta_n^2 & \theta_n^3 & \dots & \theta_n^m \end{pmatrix} = \begin{pmatrix} \theta_1^1 & -\xi\theta_1^1 & \theta_1^3 & \dots & \theta_1^m \\ \dots & \dots & \dots & \dots & \dots \\ \theta_n^1 & -\xi\theta_n^1 & \theta_n^3 & \dots & \theta_n^m \end{pmatrix} \tag{28}$$

(when $m = 2$ it has two columns). Note that by an appropriate choice of the numbers θ_j^1, $j = 1,2,\dots,n$, we can guarantee that all the elements of this matrix are linearly independent together with 1 over $\mathbb{Z}$. Then

$$\mathrm{DIM}_{\mathbb{Z}}\, \Theta = \mathrm{DIM}_{\mathbb{Z}}\, {}^t\Theta = m+n,$$

and the matrix is good. At the same time we see from (27) that the elements of the matrix (28) are algebraically dependent (so Theorem 10 below does not contradict Theorem 9 above).

Theorem 10. *Let $2 \leqslant m < n$. Then for almost all collections of $n(m-1)$ real numbers $(\theta_1^i, \theta_2^i, \dots, \theta_n^i)$ with $i = 1, 3, 4, \dots, m$ such that $1/3 < \theta_1^1, \theta_2^1, \dots, \theta_{n-1}^1 < 2/3 < \theta_n^1 < 1$ and $\theta_j^i \in (0,1)$ for $3 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ the sequence of best approximation vectors for the matrix (28) differs from the sequence of integer vectors*

$$(p_\nu, q_\nu, \underbrace{0, \dots, 0}_{m+n-2}), \qquad \nu = 1, 2, 3, \dots,$$

by at most finitely many elements. Hence $R(\Theta) = 2$.

Corollary. *In the case $m \geqslant 2$ and $n > m$ there exist matrices Θ consisting of elements linearly independent together with 1 over $\mathbb{Z}$ and such that $R(\Theta) = 2$.*

Proof of Theorem 10. We note that

$$\max_{1 \leqslant j \leqslant n} \|p_\nu \theta_j^1 + q_\nu \theta_j^2\| = \max_{1 \leqslant j \leqslant n} \|(p_\nu - q_\nu \xi)\theta_j^1\| = |(p_\nu - q_\nu \xi)\theta_n^1|.$$

On the other hand, from the theory of continued fractions (see § 3.1, formulae (42) and (43)) we see that

$$\max_{1 \leqslant j \leqslant n} \|p_\nu \theta_j^1 + q_\nu \theta_j^2\| = |(p_\nu - q_\nu \xi)\theta_n^1| < \frac{1}{q_{\nu+1}}\,. \tag{29}$$

To establish that the vector (p_ν, q_ν) is a best approximation vector for the matrix (28) and that the vector $(p_{\nu+1}, q_{\nu+1})$ is precisely the next best approximation vector for (28), it is sufficient to show that

$$\min_{(x_1,\dots,x_m;y_1,\dots,y_n)} \max_{1 \leqslant j \leqslant n} |(x_1 - \xi x_2)\theta_j^1 + x_3\theta_j^3 + \dots + x_m \theta_j^m + y_j| > |(p_\nu - q_\nu \xi)\theta_n^1|, \tag{30}$$

where the minimum is taken over all integer points $(x_1, \dots, x_m; y_1, \dots, y_n)$ such that

$$0 \neq \max_{1 \leqslant i \leqslant m} |x_i| \leqslant q_{\nu+1}, \qquad \max_{1 \leqslant j \leqslant n} |y_j| \leqslant 1 + |x_1 - \xi x_2| + |x_3| + \dots + |x_m|, \tag{31}$$

and

$$(x_1, \dots, x_m; y_1, \dots, y_n) \neq \pm(p_\nu, q_\nu, \underbrace{0, \dots, 0}_{m+n-2}), \pm(p_{\nu+1}, q_{\nu+1}, \underbrace{0, \dots, 0}_{m+n-2}).$$

If $x_3 = \dots = x_m = y_1 = \dots = y_n = 0$, then nothing depends on the properties of the numbers θ_j^i, and

$$\max_{1 \leqslant j \leqslant n} |(x_1 - \xi x_2)\theta_j^1 + x_3\theta_j^3 + \dots + x_m\theta_j^m + y_j| = |(x_1 - \xi x_2)\theta_n^1| > |(p_\nu - q_\nu \xi)\theta_n^1|,$$

since $p_\nu < q_\nu$ and since the convergents to a number correspond to the best one-dimensional approximations (see § 3.1).

The condition (30) is valid if for any integer vector $(x_1, \dots, x_m, y_1, \dots, y_n)$ under consideration satisfying (31) and such that

$$|x_3| + \dots + |x_m| + |y_1| + \dots + |y_n| \neq 0 \tag{32}$$

one has

$$(\theta_j^1,\theta_j^3,\dots,\theta_j^m)\notin J_\nu(x_1,x_2,\dots,x_m;y_j)$$

for $1\leqslant j\leqslant n$, where

$$J_\nu(x_1,x_2,\dots,x_m;y_j)=\biggl\{(\theta_j^1,\theta_j^3,\dots,\theta_j^m)\in[0,1]^{m-1}: \\ |(x_1-\xi x_2)\theta_j^1+x_3\theta_j^3+\dots+x_m\theta_j^m+y_j|\leqslant\frac{1}{q_{\nu+1}}\biggr\}.$$

For the Lebesgue measure of this set we have an upper estimate of the form

$$\mu\bigl(J_\nu(x_1,x_2,\dots,x_m;y_j)\bigr)\leqslant\frac{m^{3/2}}{q_{\nu+1}(|x_1-\xi x_2|+|x_3|+\dots+|x_m|)}\,. \tag{33}$$

Moreover, in the case $x_3=\dots=x_m=0$ we see from (32) that at least one y_j is not equal to 0. Thus, if $|x_1-\xi x_2|<1/2$, then (30) is satisfied automatically. So we can assume that

$$|x_1-\xi x_2|\geqslant\frac{1}{2}\,.$$

Therefore, for the condition (30) to hold it suffices that

$$\begin{pmatrix}\theta_1^1 & \theta_1^3 & \dots & \theta_1^m\\ \dots & \dots & \dots & \dots\\ \theta_n^1 & \theta_n^3 & \dots & \theta_n^m\end{pmatrix}\notin J_\nu=J_\nu^{(1)}\cup J_\nu^{(2)},$$

where

$$J_\nu^{(1)}=\bigcup\nolimits_1\bigcup\nolimits_2 J_\nu\bigl(x_1,x_2,\underbrace{0,\dots,0}_{m-2};y_1\bigr)\times\dots\times J_\nu\bigl(x_1,x_2,\underbrace{0,\dots,0}_{m-2};y_n\bigr),$$

$$J_\nu^{(2)}=\bigcup\nolimits_3\bigcup\nolimits_4\bigcup\nolimits_5 J_\nu(x_1,x_2,x_3,\dots,x_m;y_1)\times\dots\times J_\nu(x_1,x_2,x_3,\dots,x_m;y_n),$$

with the unions $\bigcup_i$, $i=1,\dots,5$, taken over the following sets:

$\bigcup_1$: over (x_1,x_2) such that $0<\max\{|x_1|,|x_2|\}\leqslant q_{\nu+1}$, $|x_1-\xi x_2|\geqslant\dfrac{1}{2}$;

$\bigcup_2$: over $(y_1,\dots,y_n)$ such that $\max\limits_{1\leqslant j\leqslant n}|y_j|\leqslant 4|x_1-\xi x_2|$;

$\bigcup_3$: over (x_1,x_2) such that $0<\max\{|x_1|,|x_2|\}\leqslant q_{\nu+1}$;

$\bigcup_4$: over $(x_3,\dots,x_m)$ such that $0<\max\limits_{3\leqslant i\leqslant m}|x_i|\leqslant q_{\nu+1}$;

$\bigcup_5$: over $(y_1,\dots,y_n)$ such that $\max\limits_{1\leqslant j\leqslant n}|y_j|\leqslant 2\Bigl(|x_1-\xi x_2|+\sum\limits_{i=3}^m|x_i|\Bigr)$.

It follows from (33) that

$$\mu\bigl(J_\nu^{(1)}\bigr)\ll q_{\nu+1}^{2-n},\qquad \mu\bigl(J_\nu^{(2)}\bigr)\ll q_{\nu+1}^{m-n},\qquad \mu(J_\nu)\ll q_{\nu+1}^{m-n}\leqslant q_{\nu+1}^{-1}.$$

The series $\sum_{\nu=1}^\infty 1/q_\nu$ converges. Hence, the Borel–Cantelli lemma gives us Theorem 10.

To finish this subsection we note that in the case $m = n = 2$ the author knows no example of a good matrix Θ with the elements linearly independent together with 1 over $\mathbb{Z}$ and such that $R(\Theta) = 2$.

2.3. Degeneracy of dimension, $R(\Theta) = 3$. The results of the previous subsection show to what extent the lower bound in the statement (i) of Theorem 7 in the case $m \geqslant 2$ is sharp. We show below that the lower bound of the statement (iii) of Theorem 7 is also sharp. For this purpose we shall use singular matrices.

First of all, we consider a very special case.

Suppose that $\xi^1, \xi^2 \in (0, 1/2)$ are linearly independent together with 1 over $\mathbb{Z}$. Let $(x_{1,\nu}, x_{2,\nu}, x_{3,\nu})$ be the 'extended' best approximation vectors for the matrix (ξ^1, ξ^2). (Here we use notation which does not correspond to the beginning of the present paper but is convenient here.) Then

$$\|x_{1,\nu}\xi^1 + x_{2,\nu}\xi^2\| = |x_{1,\nu}\xi^1 + x_{2,\nu}\xi^2 + x_{3,\nu}| < M_{\nu+1}^{-2},$$

where we can assume that

$$M_\nu = \max_{i=1,2,3} |x_i|.$$

Let

$$\theta_j^1 = \xi^1\theta_j^3, \quad \theta_j^2 = \xi^2\theta_j^3, \qquad 1 \leqslant j \leqslant n,$$

and consider the matrix

$$\begin{pmatrix} \theta_1^1 & \theta_1^2 & \theta_1^3 \\ \multicolumn{3}{c}{\dotfill} \\ \theta_n^1 & \theta_n^2 & \theta_n^3 \end{pmatrix} = \begin{pmatrix} \xi^1\theta_1^3 & \xi^2\theta_1^3 & \theta_1^3 \\ \multicolumn{3}{c}{\dotfill} \\ \xi^1\theta_n^3 & \xi^2\theta_n^3 & \theta_n^3 \end{pmatrix}. \tag{34}$$

Theorem 11. *Suppose that the series*

$$\sum_{\nu=1}^{\infty} M_{\nu+1}^3 \|x_{1,\nu}\xi^1 + x_{2,\nu}\xi^2\|^n \tag{35}$$

converges. Then for almost all tuples $(\theta_1^3, \dots, \theta_n^3) \in \mathbb{R}^n$ with $0 < \theta_2^3, \dots, \theta_n^3 < 1/2 < \theta_1^3 < 1$ the sequence of best approximations for the matrix Θ defined in (34) differs from the sequence

$$\bigl(x_{1,\nu}, x_{2,\nu}, x_{3,\nu}, \underbrace{0, \dots, 0}_{n}\bigr)$$

by at most finitely many elements.

The proof of Theorem 11 is quite similar to the proof of Theorem 10. One should ensure the inequality

$$\min_{x_1,x_2,x_3;y_1,\dots,y_n} \max_{1 \leqslant j \leqslant n} |\theta_j^3(x_1\xi^1 + x_2\xi^2 + x_3) + y_j| > |x_{1,\nu}\xi^1 + x_{2,\nu}\xi^2 + x_{3,\nu}|$$

for almost all $(\theta_1^3, \dots, \theta_n^3) \in \mathbb{R}^n$, where the minimum is taken over all integer vectors $(x_1, x_2, x_3, y_1, \dots, y_n)$ different from the vectors

$$\pm\bigl(x_{1,\nu}, x_{2,\nu}, x_{3,\nu}, \underbrace{0, \dots, 0}_{n}\bigr), \quad \pm\bigl(x_{1,\nu+1}, x_{2,\nu+1}, x_{3,\nu+1}, \underbrace{0, \dots, 0}_{n}\bigr)$$

and such that

$$0 \neq \max_{i=1,2,3} |x_i| \leqslant M_{\nu+1},$$

Of course, one should distinguish the cases $\mathbf{y} = \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$.

Since the series $\sum_\nu M_\nu^{-1}$ converges (this follows from the exponential rate of growth of M_ν; see, for example, the lemma in [34]), we get the following statement.

Corollary. *Let $n \geqslant 2$. Then $R(\Theta) = 3$ for the matrix (34) for almost all tuples $(\theta_1^3, \dots, \theta_n^3) \in \mathbb{R}^n$.*

We now prove a generalization of a recent result of Moshchevitin and German in [34]. This result itself is a generalization of the author's result in [35]. We note that in [34] some other results related to the existence of singular systems of a special kind (with $n = 1$) were announced, whose complete proofs have not yet been published. One of these will be stated below in § 12 (Theorem 69).

Let $m^* > m$. For a matrix Θ of the form (1) we consider an 'extended' matrix

$$\Theta^* = \begin{pmatrix} \theta_1^1 & \dots & \theta_1^m & \theta_1^{m+1} & \dots & \theta_1^{m^*} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \theta_n^1 & \dots & \theta_n^m & \theta_n^{m+1} & \dots & \theta_n^{m^*} \end{pmatrix}$$

with real elements. We have enlarged the matrix Θ by adding $n(m^* - m)$ elements. The collection of additional elements can be identified with a vector $\underline{\Theta} = (\theta_1^{m+1}, \dots, \theta_n^{m^*})$ in $\mathbb{R}^{n(m^*-m)}$. The following statement is a direct generalization of a result in [34]. It develops ideas in [35] and [33].

Theorem 12. *Let*

$$\mathbf{z}_\nu = (x_{1,\nu}, \dots, x_{m,\nu}, y_{1,\nu}, \dots, y_{n,\nu}) \in \mathbb{Z}^{m+n}, \qquad \nu = 1, 2, 3, \dots,$$

be all the best approximations for the matrix Θ. Let M_ν and ζ_ν be the elements of the sequences (10) and (11). Consider the integer vectors

$$\mathbf{z}_\nu^* = \bigl(x_{1,\nu}, \dots, x_{m,\nu}, \underbrace{0, \dots, 0}_{m^*-m}, y_{1,\nu}, \dots, y_{n,\nu}\bigr) \in \mathbb{Z}^{m^*+n}. \tag{36}$$

Assume that the series

$$\sum_{\nu=1}^{\infty} M_{\nu+1}^{\max(m+n,m^*)} (\log M_{\nu+1})^{\delta(m^*,m+n)} \zeta_\nu^n, \qquad \delta(a,b) = \begin{cases} 1, & a = b, \\ 0, & a \neq b, \end{cases} \tag{37}$$

converges. Then for almost all 'additional' vectors $\underline{\Theta} \in \mathbb{R}^{n(m^*-m)}$ (in the sense of Lebesgue measure) the sequence of best approximations for the matrix Θ^* differs from the sequence $\mathbf{z}_\nu^*$ by at most finitely many elements.

From Theorem 12 and Proposition 1 we deduce the following statement.

Corollary. *Let $m \geqslant 2$ and let Θ be a ψ-singular matrix. Suppose that the series*

$$\sum_{\nu=1}^{\infty} M_\nu^{\max(m+n,m^*)} (\log M_\nu)^{\delta(m^*,m+n)} (\psi(M_\nu))^n \tag{38}$$

converges. Then for almost all 'additional' vectors $\underline{\Theta} \in \mathbb{R}^{n(m^*-m)}$ *the sequence of best approximations for the matrix* Θ^* *differs from the sequence* (36) *by at most finitely many elements.*

Let $n = 1$, $m = 2$, and $\psi(t) = t^{-m^*-2}$. By taking θ_j^1, θ_j^2 as in Theorem 1 the previous corollary transforms into the following statement in the papers [35] and [33].

Theorem 13. *Let* $n = 1$. *Then for any* $m \geqslant 2$ *there exists a vector* Θ *with algebraically independent elements* θ^i $(1 \leqslant i \leqslant m)$ *such that all the best approximation vectors* $\mathbf{z}_\nu$ *with sufficiently large* ν *belong to some three-dimensional subspace, and so* $R(\Theta) = 3$.

By taking arbitrary n and $m \geqslant 3$ Theorems 11 and 12 lead to the following result.

Theorem 14. *Let* $m \geqslant 3$. *Then for any* n *there exists a matrix* Θ *consisting of elements* θ_j^i *which are linearly independent together with* 1 *over* $\mathbb{Z}$ *and such that all the best approximation vectors* $\mathbf{z}_\nu$ *with* ν *sufficiently large belong to some three-dimensional subspace, and so* $R(\Theta) = 3$.

*Proof of Theorem*12. One can assume that $\theta_j^i \in [0, 1]$ for all i, j.

We must prove that under the conditions of our theorem for a given matrix Θ and for almost all 'additional' vectors $\underline{\Theta}$ there exists a ν_0 such that for all $\nu \geqslant \nu_0$

$$\min \max_{1\leqslant j\leqslant n} \left| \sum_{1\leqslant i\leqslant m^*} x_i\theta_j^i + y_j \right| > \zeta_\nu, \tag{39}$$

where the minimum is taken over all integer points

$$\mathbf{z} = (x_1, \dots, x_{m^*}, y_{1,\nu}, \dots, y_n) \in \mathbb{Z}^{m^*+n} \setminus \{\mathbf{0}\}$$

such that

$$\max_{1\leqslant j\leqslant m^*} |x_j| \leqslant M_{\nu+1}, \qquad \mathbf{z} \neq \mathbf{z}_\nu^*.$$

The condition (39) is satisfied if for any vector

$$(x_1, \dots, x_{m^*}, y_1, \dots, y_n) \in \mathbb{Z}^{m^*+n} \setminus \{\mathbf{0}, \pm\mathbf{z}_\nu, \pm\mathbf{z}_{\nu+1}\}$$

with $\max_{1\leqslant i\leqslant m^*} |x_i| \leqslant M_{\nu+1}$ there exists a j with $1 \leqslant j \leqslant n$ such that

$$x_{m+1}\theta_j^{m+1} + \dots + x_{m^*}\theta_j^{m^*} \notin J_\nu(y_j, x_1, \dots, x_m), \tag{40}$$

where

$$J_\nu(y_j, x_1, \dots, x_m) = (-y_j - x_1\theta_j^1 - \dots - x_m\theta_j^m - \zeta_\nu, -y_j - x_1\theta_j^1 - \dots - x_m\theta_j^m + \zeta_\nu).$$

The condition (40) means that the distance from the point $(\theta_j^{m+1},\dots,\theta_j^{m^*})\in[0,1]^{m^*-m}$ to the subspace

$$\{(u_{m+1},\dots,u_{m^*})\in\mathbb{R}^{m^*-m}:\ x_{m+1}u_{m+1}+\dots+x_{m^*}u_{m^*}=-y_j-x_1\theta_j^1-\dots-x_m\theta_j^m\}$$

is not less than $\zeta_\nu\cdot(x_{m+1}^2+\dots+x_{m^*}^2)^{-1/2}$.

Let

$$\Omega_\nu(\mathbf{z})=\Omega_\nu(\mathbf{x},\mathbf{y})=\{\underline{\Theta}\in[0,1]^{n(m^*-m)}:\\ x_{m+1}\theta_j^{m+1}+\dots+x_{m^*}\theta_j^{m^*}\notin J_\nu(y_j,x_1,\dots,x_m),\ 1\leqslant j\leqslant n\}$$

and let

$$\Omega_\nu=\bigcup_{\mathbf{y}}\bigcup_{\mathbf{x}}\Omega_\nu(\mathbf{x},\mathbf{y}).$$

In the last formula the unions are taken over all integer points in the sets

$$\Bigl\{\mathbf{y}:\max_{1\leqslant j\leqslant n}|y_j|\leqslant(m^*+1)M_{\nu+1}\Bigr\},\qquad\Bigl\{\mathbf{x}:0<\max_{1\leqslant i\leqslant m^*}|x_i|\leqslant M_{\nu+1}\Bigr\}.$$

By the Borel–Cantelli lemma our theorem is true if

$$\sum_{\nu\geqslant\nu_0}\mu(\Omega_\nu)\to0,\qquad\nu_0\to+\infty.\tag{41}$$

But

$$\mu(\Omega_\nu)\ll\zeta_\nu^nM_{\nu+1}^n\sum_{x_1,\dots,x_m}\ \sum_{x_{m+1},\dots,x_{m^*}}\frac{1}{(\max_{m\leqslant i\leqslant m^*}|x_i|)^n}\\ \ll\zeta_\nu^nM_{\nu+1}^{n+m}\sum_{1\leqslant t\leqslant M_{\nu+1}}t^{m^*-m-n-1}\ll\zeta_\nu^nM_{\nu+1}^{\max(m+n,m^*)}(\log M_{\nu+1})^{\delta(m^*,m+n)}.$$

Now (41) follows from the convergence of the series (37), and the theorem is proved.

3. One-dimensional Diophantine approximation

Here we discuss the simplest case $m=n=1$. In this case we deal with the problem of approximating one number $\alpha=\theta_1^1$ by rational fractions.

3.1. Continued fractions. It is a well-known fact that the problem of investigating the best approximations to one real number admits a solution in terms of continued fractions (see [36]). Recall that for a representation of a real number α as a continued fraction

$$\alpha=[a_0;a_1,a_2,\dots,a_t,\dots]=a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\dots+\cfrac{1}{a_t+\dots}}},$$

$a_0 \in \mathbb{Z},\ a_t \in \mathbb{N},\ t = 1, 2, 3, \dots$ (this fraction is finite or infinite according to whether α is rational or not), the convergents are defined to be the rational fractions

$$\frac{p_\nu}{q_\nu} = [a_0; a_1, a_2, \dots, a_\nu] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots + \cfrac{1}{a_\nu}}}\,.$$

The best approximations defined in § 1.3 (more precisely, the 'extended' vectors $\mathbf{z}_\nu \in \mathbb{Z}^2$) coincide with the best approximations *of the second kind* (in the terminology of the book [36]). The following statement is valid (see [36], Theorem 16).

Theorem 15. *All the best approximations $\mathbf{z}_\nu$ are of the form $\mathbf{z}_\nu = (q_\nu, p_\nu)$, where q_ν and p_ν are the denominator and the numerator of some convergent to α.*

We note that if $\alpha \neq p_\nu/q_\nu$ (for example, if α is irrational), then the following inequalities are valid (see [36], Theorems 9 and 13):

$$\frac{1}{q_\nu(q_\nu + q_{\nu+1})} < \left|\alpha - \frac{p_\nu}{q_\nu}\right| < \frac{1}{q_\nu q_{\nu+1}}\,. \tag{42}$$

In particular,

$$\|q_\nu \alpha\| > \frac{1}{2q_{\nu+1}}\,. \tag{43}$$

In terms of the notation in § 1.3 (for $\alpha \in (0,1)$), we have

$$\zeta_\nu \geqslant \frac{1}{2M_{\nu+1}} \qquad \forall\, \nu \in \mathbb{N}. \tag{44}$$

Here we note that for the difference in (42) there is a simple and elegant equality

$$\left|\alpha - \frac{p_\nu}{q_\nu}\right| = \frac{1}{q_\nu^2(\alpha_{\nu+1} + \alpha_\nu^*)}\,, \tag{45}$$

where

$$\alpha_{\nu+1} = [a_{\nu+1}; a_{\nu+2}, a_{\nu+3}, \dots], \qquad \alpha_\nu^* = [0; a_\nu, \dots, a_1].$$

This equality makes possible a very detailed investigation of one-dimensional Diophantine approximation. For example, it allows us to reduce the study of the *Lagrange spectrum*

$$\mathbb{L} = \left\{\lambda \in \mathbb{R} :\ \text{there is an } \alpha \in \mathbb{R} \text{ such that } \lambda = \Bigl(\liminf_{q\to+\infty} q\|q\alpha\|\Bigr)^{-1}\right\}$$

to certain problems relating to doubly infinite sequences [37]. There are many papers devoted to the Lagrange spectrum $\mathbb{L}$ (see the bibliography in the book [37] and the wonderful survey [38]). Apparently, no multidimensional generalization of the equality (45) is known.

Suppose that an irrational number α forms a singular system with $m = n = 1$. Then from Proposition 1 we see that $\zeta_\nu < 1/(2M_{\nu+1})$ for sufficiently large ν.

This contradicts (44). Hence, there are no irrational numbers θ which are singular systems.

One can rewrite the equality (45) in the form

$$q_\nu \|q_\nu \alpha\| = \frac{1}{\alpha_{\nu+1} + \alpha_\nu^*}\,.$$

Here we recall a similar equality

$$q_{\nu+1} \|q_\nu \alpha\| = \frac{1}{1 + \dfrac{1}{\alpha_{\nu+2}\alpha_{\nu+1}^{**}}}\,, \tag{46}$$

where

$$\alpha_{\nu+1}^{**} = a_{\nu+1} + \alpha_\nu^* = [a_{\nu+1}; a_\nu, \dots, a_1]$$

(see, for example, [39]).

In particular, this equality enables one to study the *Dirichlet spectrum*

$$\mathbb{D} = \left\{ \lambda \in \mathbb{R} : \text{ there is an } \alpha \in \mathbb{R} \text{ such that } \lambda = \left(\limsup_{q\to+\infty} q \min_{x\leqslant q} \|x\alpha\| \right)^{-1} \right\}.$$

There are few papers devoted to the study of the Dirichlet spectrum (see [40] and the bibliography there).

Of course, it is possible to give the definitions of the Lagrange and Dirichlet spectra in terms of the function $\psi_\alpha(t)$.

3.2. The function $\boldsymbol{\psi_\alpha(t)}$. For $m = n = 1$ and $\theta_1^1 = \alpha$ the function (20) becomes

$$\psi_\alpha(t) = \min_{1\leqslant x\leqslant t} \|\alpha x\|.$$

Recently Kan and Moshchevitin [41] obtained the following result.

Theorem 16. *Consider two irrational numbers α and β with $\alpha \pm \beta \notin \mathbb{Z}$. Then the difference function*

$$\psi_\alpha(t) - \psi_\beta(t)$$

changes sign infinitely often as $t \to +\infty$.

We do not give a proof. The only thing we would like to say here is that the proof involves the formulae (45) and (46).

We also note that Khintchine's Theorems 1 and 2 in § 1.2 show that the result of Theorem 16 cannot be generalized to higher dimensions.

In Theorem 1 one can take $\psi(t) = o(t^{-2}),\ t \to +\infty$. Let θ^1 and θ^2 be the numbers whose existence is asserted there. We take (β^1, β^2) to be badly approximable numbers (in the sense of a linear form):

$$\inf_{(x_1,x_2)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\|x_1\beta^1 + x_2\beta^2\| (\max\{|x_1|, |x_2|\})^2 \right) > 0.$$

Then for sufficiently large t

$$\psi_{(\theta^1,\theta^2)}(t) < \psi_{(\beta^1,\beta^2)}(t).$$

The situation for simultaneous approximations is quite similar. We use Theorem 2 with a function satisfying $\psi_1(t) = o(t^{-1/2})$ for $t \to +\infty$. Thus, we obtain θ_1 and θ_2 from Theorem 2. Also, we must take simultaneously badly approximable numbers $\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$:

$$\inf_{x\in\mathbb{Z}\setminus\{0\}}\Bigl(\max_{j=1,2}\|x\beta_j\|\cdot|x|^{1/2}\Bigr)>0.$$

Then

$$\psi_{\left(\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right)}(t)<\psi_{\left(\begin{smallmatrix}\beta_1\\ \beta_2\end{smallmatrix}\right)}(t)$$

for sufficiently large t.

3.3. Two-dimensional lattices. Here we generalize the inequalities (42) (and hence the lower bound in (44)) in a way convenient for our purposes. Consider a two-dimensional completely rational linear subspace $\pi\subset\mathbb{R}^d$ and the two-dimensional lattice $\Lambda=\pi\cap\mathbb{Z}^d$. Suppose that the set

$$\mathscr{B}=\pi\cap\Bigl\{\mathbf{z}=(\mathbf{x},\mathbf{y}):\max_{1\leqslant i\leqslant m}|x_i|\leqslant 1\Bigr\}$$

is compact. Then the sup-norm in the space of vectors $\mathbf{x}$ induces a certain norm $|\cdot|_\bullet$ on the subspace π (so that the 'unit ball' $\mathscr{B}=\{\mathbf{z}\in\pi:|\mathbf{z}|_\bullet=1\}$ is a bounded convex $\mathbf{0}$-symmetric set). Consider a one-dimensional subspace $\ell\subset\pi$. Suppose that $\mathbf{0}$ is the only point of Λ which belongs to ℓ, and moreover, that

$$\ell\cap\{\mathbf{z}=(\mathbf{x}',\mathbf{y}'):x_1'=\dots=x_m'=0\}=\{\mathbf{0}\}.$$

Take a linear subspace $\mathscr{L}$ such that

$$\mathscr{L}\cap\pi=\ell,\qquad\dim\mathscr{L}\geqslant 1.$$

Since ℓ is not a completely rational subspace, $\mathscr{L}$ is also not completely rational.

We now define a best approximation to the subspace ℓ by points of the lattice Λ with respect to the norm $|\cdot|_\bullet$ and the subspace $\mathscr{L}$ to be a point $\mathbf{z}=(\mathbf{x},\mathbf{y})\in\Lambda$ such that

$$\operatorname{dist}(\mathbf{z},\mathscr{L})=\min\operatorname{dist}(\mathbf{z}',\mathscr{L}),$$

where the minimum is taken over all points $\mathbf{z}'\in\Lambda$ with

$$0<|\mathbf{z}'|_\bullet\leqslant|\mathbf{z}|_\bullet,$$

and $\operatorname{dist}(\mathbf{z},\mathscr{L})$ stands for the distance from $\mathbf{z}$ to the subspace $\mathscr{L}$ in the sup-norm in $\mathbb{R}^d$.

The best approximations naturally form infinite sequences

$$\pm\mathbf{z}_1,\pm\mathbf{z}_2,\dots,\pm\mathbf{z}_\nu,\pm\mathbf{z}_{\nu+1},\dots,$$
$$\operatorname{dist}(\mathbf{z}_1,\mathscr{L})>\operatorname{dist}(\mathbf{z}_2,\mathscr{L})>\dots>\operatorname{dist}(\mathbf{z}_\nu,\mathscr{L})>\operatorname{dist}(\mathbf{z}_{\nu+1},\mathscr{L})>\cdots,$$
$$|\mathbf{z}_1|_\bullet<|\mathbf{z}_2|_\bullet<\dots<|\mathbf{z}_\nu|_\bullet<|\mathbf{z}_{\nu+1}|_\bullet<\cdots.$$

These sequences are analogous to the sequences (10) and (11). From the construction it follows that the quantities $\mathrm{dist}(\mathbf{z}_\nu, \mathscr{L})$, $|\mathbf{z}_\nu|_\bullet$ change strictly monotonically. We say that the triple $(\pi, \ell, \mathscr{L})$ is *good* if for sufficiently large ν the vector $\mathbf{z}_\nu \in \Lambda$ is uniquely determined up to the sign.

An obvious generalization of the inequalities (42) is as follows.

Proposition 2. *Suppose that the triple $(\pi, \ell, \mathscr{L})$ is good. Then there exist positive constants C_1 and C_2 depending on π, ℓ, and $\mathscr{L}$ such that for all ν*

$$C_1 \leqslant \mathrm{dist}(\mathbf{z}_\nu, \mathscr{L})|\mathbf{z}_{\nu+1}|_\bullet \leqslant C_2.$$

To verify Proposition 2 it is sufficient to consider the two-dimensional convex $\mathbf{0}$-symmetric set

$$\{\mathbf{z} \in \pi : \ |\mathbf{z}|_\bullet \leqslant |\mathbf{z}_{\nu+1}|_\bullet, \ \mathrm{dist}(\mathbf{z}, \mathscr{L}) \leqslant \mathrm{dist}(\mathbf{z}_\nu, \mathscr{L})\}.$$

The linearly independent points $\mathbf{z}_\nu, \mathbf{z}_{\nu+1} \in \Lambda$ belong to its boundary.

4. Singular systems in simultaneous Diophantine approximation

The case $m = 1$ is known as simultaneous Diophantine approximation. Everywhere in this section we suppose that $m = 1$, and we use the notation $\theta_j^1 = \theta_j$, $1 \leqslant j \leqslant n$. In this case

$$\mathbf{x} = x_1 = x \in \mathbb{Z}^1, \qquad M(\mathbf{x}) = |x|, \qquad L_j(x) = \theta_j x.$$

Thus, we are interested in small values of

$$\max_{1\leqslant j\leqslant n} \|x\theta_j\|.$$

The empty parallelepipeds (16), (17) take the form

$$\begin{aligned}
\Omega_\nu^{(0)} &= \Big\{\mathbf{z}' = (x', y_1', \dots, y_n') : \ |x'| \leqslant x_\nu, \ \max_{1\leqslant j\leqslant n} |x'\theta_j + y_j'| \leqslant \max_{1\leqslant j\leqslant n} \|x_\nu\theta_j\|\Big\}, \\
\Omega_\nu &= \Big\{\mathbf{z}' = (x', y_1', \dots, y_n') : \ |x'| \leqslant x_{\nu+1}, \ \max_{1\leqslant j\leqslant n} |x'\theta_j + y_j'| \leqslant \max_{1\leqslant j\leqslant n} \|x_\nu\theta_j\|\Big\}.
\end{aligned} \tag{47}$$

In the first parallelepiped there are no integer points except for $\mathbf{0}$ and $\pm\mathbf{z}_\nu$. In the second parallelepiped there are no integer points except for $\mathbf{0}$ and $\pm\mathbf{z}_\nu$, $\pm\mathbf{z}_{\nu+1}$. Here $\mathbf{z}_\nu = (x_\nu, y_{1,\nu}, \dots, y_{n,\nu})$ and $x_\nu > 0$.

Take a function $\psi(t) = o(t^{-1/n})$. The set of real numbers $\Theta \in \mathbb{R}^n$ forms a ψ-singular system if for sufficiently large t the Diophantine system

$$\max_{1\leqslant j\leqslant n} \|x\theta_j\| \leqslant \psi(t), \qquad 0 < x \leqslant t,$$

admits an integer solution $x \in \mathbb{Z}$. In other words, this means that for the denominators x_ν of the best approximations one has

$$\max_{1\leqslant j\leqslant n} \|x_\nu\theta_j\| \leqslant \psi(x_{\nu+1})$$

for sufficiently large ν.

Recall that for $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ we denote by $\dim_{\mathbb{Z}} \Theta$ the maximal number of numbers linearly independent over $\mathbb{Z}$ in the set $1, \theta_1, \dots, \theta_n$.

4.1. Linear independence of best approximation vectors. We formulate immediate corollaries of the above general results.

Proposition 2 in § 3.3 gives us the following result.

Corollary 1. *Suppose that* $\dim_{\mathbb{Z}} \Theta = 2$. *Then there exists a constant* $\gamma = \gamma(\Theta) > 0$ *such that*

$$\zeta_\nu \geqslant \frac{\gamma}{M_{\nu+1}} \qquad \forall\, \nu \in \mathbb{N}.$$

From the statement (ii) of Theorem 7 and the remark in § 2.1 (see p. 444) we deduce the following corollaries.

Corollary 2. *If* $n \geqslant 2$ *and* $\dim_{\mathbb{Z}} \Theta \geqslant 3$, *then there exist infinitely many* ν *such that the vectors* $\mathbf{z}_\nu$, $\mathbf{z}_{\nu+1}$, *and* $\mathbf{z}_{\nu+2}$ *are linearly independent.*

Corollary 3. *Suppose that* $n = 2$ *and* $\dim_{\mathbb{Z}} \Theta = 3$. *Then there exist infinitely many* ν *such that for the best approximation vectors* $z_{\nu+i} = (x_{\nu+i}, y_{1,\nu+i}, y_{2,\nu+i})$ $(i = 0, 1, 2)$ *one has*

$$\det \begin{pmatrix} x_\nu & y_{1,\nu} & y_{2,\nu} \\ x_{\nu+1} & y_{1,\nu+1} & y_{2,\nu+1} \\ x_{\nu+2} & y_{1,\nu+2} & y_{2,\nu+2} \end{pmatrix} \neq 0.$$

Corollary 3 can be found in Lagarias' paper [30]. The following more general statement can be found in implicit form in Jarník's paper [13] (pp. 333–337).

Corollary 4. *Suppose that* $n \geqslant 2$ *and* $\dim_{\mathbb{Z}} \Theta \geqslant 3$. *Then there exist two indices* $j_1 \neq j_2$ *and an infinite sequence of values of* ν *such that for the best approximation vectors* $z_{\nu+i} = (x_{\nu+i}, y_{1,\nu+i}, \dots, y_{n,\nu+i})$ $(i = 0, 1, 2)$ *one has*

$$\det \begin{pmatrix} x_\nu & y_{j_1,\nu} & y_{j_2,\nu} \\ x_{\nu+1} & y_{j_1,\nu+1} & y_{j_2,\nu+1} \\ x_{\nu+2} & y_{j_1,\nu+2} & y_{j_2,\nu+2} \end{pmatrix} \neq 0. \tag{48}$$

Formally speaking, the statement in [13] is somewhat weaker. Here we would like to make some comments on the proof of Corollary 4 (cf. the lemma on p. 333 in [13]).

By Corollary 2 there are infinitely many ν such that the rank of the matrix

$$\begin{pmatrix} x_\nu & y_{1,\nu} & \dots & y_{n,\nu} \\ x_{\nu+1} & y_{1,\nu+1} & \dots & y_{n,\nu+1} \\ x_{\nu+2} & y_{1,\nu+2} & \dots & y_{n,\nu+2} \end{pmatrix} \tag{49}$$

is equal to 3. If all the determinants (48) are equal to zero, then for some j_1, j_2, j_3 we have

$$\det \begin{pmatrix} y_{j_1,\nu} & y_{j_2,\nu} & y_{j_3,\nu} \\ y_{j_1,\nu+1} & y_{j_2,\nu+1} & y_{j_3,\nu+1} \\ y_{j_1,\nu+2} & y_{j_2,\nu+2} & y_{j_3,\nu+1} \end{pmatrix} \neq 0.$$

Therefore, the three-dimensional vectors (columns of the matrix (49))

$$\begin{pmatrix} y_{j_1,\nu} \\ y_{j_1,\nu+1} \\ y_{j_1,\nu+2} \end{pmatrix}, \quad \begin{pmatrix} y_{j_2,\nu} \\ y_{j_2,\nu+1} \\ y_{j_2,\nu+2} \end{pmatrix}, \quad \begin{pmatrix} y_{j_3,\nu} \\ y_{j_3,\nu+1} \\ y_{j_3,\nu+2} \end{pmatrix} \tag{50}$$

are linearly dependent. But then the vector

$$\begin{pmatrix} x_\nu \\ x_{\nu+1} \\ x_{\nu+2} \end{pmatrix}$$

is linearly dependent with any two vectors in (50). This is not possible.

Before stating Corollary 5 we would like to do the following. Suppose that $\dim_{\mathbb{Z}} \Theta \geqslant 3$. We consider the determinant in Corollary 4 which is non-zero for some ν. Then

$$\begin{aligned} 1 &\leqslant \left| \det \begin{pmatrix} x_\nu & y_{j_1,\nu} & y_{j_2,\nu} \\ x_{\nu+1} & y_{j_1,\nu+1} & y_{j_2,\nu+1} \\ x_{\nu+2} & y_{j_1,\nu+2} & y_{j_2,\nu+2} \end{pmatrix} \right| \\ &= \left| \det \begin{pmatrix} x_\nu & y_{j_1,\nu} - \theta_{j_1} x_\nu & y_{j_2,\nu} - \theta_{j_2} x_\nu \\ x_{\nu+1} & y_{j_1,\nu+1} - \theta_{j_1} x_{\nu+1} & y_{j_2,\nu+1} - \theta_{j_2} x_{\nu+1} \\ x_{\nu+2} & y_{j_1,\nu+2} - \theta_{j_1} x_{\nu+2} & y_{j_2,\nu+2} - \theta_{j_2} x_{\nu+2} \end{pmatrix} \right| \\ &\leqslant 6 x_{\nu+2} \max_{1\leqslant j\leqslant n} \|x_\nu \theta_j\| \max_{1\leqslant j\leqslant n} \|x_{\nu+1}\theta_j\| = 6 M_{\nu+2}\zeta_\nu \zeta_{\nu+1}. \end{aligned} \tag{51}$$

Thus, for infinitely many ν

$$M_{\nu+2}\zeta_{\nu+1} \geqslant \frac{1}{6\zeta_\nu}\,.$$

The next corollary is now obvious.

Corollary 5. *Suppose that* $\dim_{\mathbb{Z}} \Theta \geqslant 3$. *Then*

$$\limsup_{\nu\to+\infty} M_{\nu+1}\zeta_\nu = +\infty. \tag{52}$$

This was proved by Jarník for $n = 2$ in [9] (Satz 9). In the multidimensional case it appears in the paper [14]. We note that in [9] the proof of (52) relies on transference principle arguments. More precisely, Jarník proves that

$$\limsup_{t\to+\infty} t\psi_\Theta(t) = +\infty, \tag{53}$$

which is a consequence of (52). Here we formulate a slightly more precise result due the author [42] (there is no reference to Jarník's paper in [42] since at that time the author was not familiar with his results).

Theorem 17. *Suppose that* $n \geqslant 2$ *and* $\dim_{\mathbb{Z}} \Theta \geqslant 3$. *Then*

$$\zeta_\nu M_{\nu+1} \to +\infty, \qquad \nu \to +\infty. \tag{54}$$

Proof. 1. First, we consider a two-dimensional lattice $\Lambda^2 \subset \mathbb{Z}^{n+1}$. Let $\det_2 \Lambda^2$ be its fundamental two-dimensional volume. Then the collection of lattices

$$\{\Lambda^2 \subset \mathbb{Z}^{n+1} : \det_2 \Lambda^2 \leqslant \gamma\}$$

is finite for any positive number γ.

2. Second, we consider the two-dimensional lattice $\Lambda_\nu^2 = \langle \mathbf{z}_\nu, \mathbf{z}_{\nu+1}\rangle_{\mathbb{Z}}$ generated by two consecutive best approximations. Since $\operatorname{conv}(\mathbf{0}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1}) \subset \Omega_\nu$ (Ω_ν is defined in (47)), it follows that for some positive constant $C(\Theta)$

$$\frac{1}{2}\det_2 \Lambda_\nu^2 = \operatorname{vol}_2\bigl(\operatorname{conv}(\mathbf{0}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1})\bigr) \leqslant C(\Theta)\zeta_\nu M_{\nu+1}.$$

3. Third, we note that from the condition $\dim_{\mathbb{Z}} \Theta \geqslant 3$ it follows that for any fixed μ we have $\mathbf{z}_\nu \notin \Lambda_\mu^2$ for sufficiently large ν. Hence, for any μ and sufficiently large ν the two-dimensional lattice Λ_ν^2 does not coincide with any of the lattices $\Lambda_1^2, \dots, \Lambda_\mu^2$.

Theorem 17 now follows.

We state here a corollary of Theorem 17 which is in some sense analogous to Proposition 2 in § 3.3.

In analogy to what was done in § 3.3, we consider a completely rational linear subspace $\pi \subset \mathbb{R}^d$ with $\dim \pi \geqslant 3$ such that $\pi \cap \{\mathbf{z} = (\mathbf{x}, \mathbf{y}) : \max_{1\leqslant i\leqslant m} |x_i| \leqslant 1\}$ is a bounded set. Consider the lattice $\Lambda = \pi \cap \mathbb{Z}^d$. Let $\ell \subset \pi$ be a one-dimensional linear subspace of π. We now suppose that ℓ is not contained in a proper completely rational subspace of π. (This condition is stronger than the condition that $\mathbf{0}$ is the only point of Λ in ℓ; here is a difference from the arguments in § 3.3)

For the induced norm $|\cdot|_\bullet$ in the subspace π we consider the sequence of best approximations $\mathbf{z}_\nu \in \Lambda$ to the subspace ℓ with respect to the norm $|\cdot|_\bullet$ and with respect to some subspace $\mathscr{L} \supset \ell = \mathscr{L} \cap \pi$. Here the definition of a good triple $(\pi, \ell, \mathscr{L})$ is quite similar to the definition in § 3.3.

Proposition 3. *Under the described conditions, for a good triple $(\pi, \ell, \mathscr{L})$*

$$\operatorname{dist}(\mathbf{z}_\nu, \mathscr{L})|\mathbf{z}_{\nu+1}|_\bullet \to +\infty, \qquad \nu \to +\infty.$$

4.2. Degeneracy of dimension for best simultaneous approximations. We give a result obtained by the author in [32] (see also [33]) which provides a counterexample to a conjecture of Lagarias in [30].

Theorem 18. *Suppose that $n \geqslant 3$. Then there exist real numbers $\theta_1, \dots, \theta_n$ that are linearly independent together with 1 over $\mathbb{Z}$ and have the property that among any $n+1$ consecutive best approximation vectors $\mathbf{z}_\nu, \dots, \mathbf{z}_{\nu+n}$ there exist at most three linearly independent vectors.*

We make a brief remark about the proof of this theorem. The numbers $\theta_1, \dots, \theta_n$ will be determined by rational approximations of them. These rational approximations themselves are constructed by a certain inductive procedure. The main tool of this inductive procedure is the following statement.

Lemma 1. *Let $\alpha = (a_1/q, \dots, a_n/q)$ be a rational point with* g.c.d.$(a_1, \dots, a_n, q) = 1$ *such that the following conditions hold.*

(i) *The finite sequence of best simultaneous approximations*

$$\mathbf{z}_\nu = (x_\nu, y_{1,\nu}, \dots, y_{n,\nu}), \qquad 1 \leqslant \nu \leqslant k,$$
$$(x_k, y_{1,k}, \dots, y_{n,k}) = (q, a_1, \dots, a_n),$$

is uniquely determined.

(ii) *There exist a two-dimensional completely rational subspace* π *and an index* $\mu<k$ *such that*

$$\mathbf{z}_\nu \in \pi, \qquad \mu \leqslant \nu \leqslant k.$$

(iii) *For the* k*th best approximation,*

$$\zeta_k < \rho(\pi)$$

($\rho(\pi)$ *is defined in* (24)).

Then there exists an $\varepsilon>0$ *such that for all points* β *in the* ε*-neighbourhood of* α *the following assertions are true.*

(i*) *All the vectors of best approximation to* α *are also vectors of best approximation to* β.

(ii*) *All the extended vectors of best approximation to* β *with denominator* x *in the interval* $x_\mu \leqslant x \leqslant x_k$ *belong to the subspace* π.

To carry out the inductive step of the induction, one must take a two-dimensional completely rational subspace $\pi_1 \ni \alpha$, and then a new rational point $\alpha_1 \in \pi_1$ close to α and such that α_1 satisfies the assertion of Lemma 1 with π replaced by π_1. Moreover, one should take α_1 in such a way that it has $>n+1$ consecutive best approximation vectors in the subspace π_1. To ensure the linear independence together with 1 over $\mathbb{Z}$ of the components of the limit vector, one should take the two-dimensional subspaces so that $R(\Theta)=n+1$.

5. Singularity and Diophantine type

Suppose that the function $\varphi(t)$ decreases to zero as $t\to+\infty$. We say that a matrix Θ is of Diophantine type $\leqslant \varphi(t)$ if for infinitely many $\mathbf{x}\in\mathbb{Z}^m$

$$\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\| \leqslant \varphi\Bigl(\max_{1\leqslant i\leqslant m}|x_i|\Bigr).$$

Jarník [13] obtained simple and beautiful bounds for the Diophantine type for singular matrices. Here we present his theorems. Laurent [43] showed that in the cases $(m,n)=(1,2),(2,1)$ Jarník's bounds cannot be improved. We shall give the exact formulations of Laurent's results in § 8.3. In this section we give Jarník's original results in §§ 5.1, 5.3, 5.5. In §§ 5.2, 5.4, 5.6 we give improvements of his results which were obtained by the author in the preprint [44].

5.1. Case $m=1$. First of all we consider the problem of simultaneous approximations. Jarník (see [13], Theorem 3) obtained the following result.

Theorem 19. *Let* $\psi(t)$ *be a continuous function of* t, *decreasing to zero as* $t\to+\infty$. *Suppose that the function* $t\psi(t)$ *increases to infinity as* $t\to+\infty$. *Let* $\omega(t)$ *be the inverse function of the function* $t\psi(t)$, *and let*

$$\varphi^{[\psi]}(t)=\psi\biggl(\omega\biggl(\frac{1}{6\psi(t)}\biggr)\biggr).$$

Let $n\geqslant 2$ *and* $\dim_{\mathbb{Z}}\Theta\geqslant 3$. *Assume that the matrix* Θ *is* ψ*-singular. Then there are infinitely many integers* x *such that*

$$\max_{1\leqslant j\leqslant n}\|x\theta_j\| \leqslant \varphi^{[\psi]}(x).$$

We now prove a more precise statement: *if the best approximations* $\mathbf{z}_\nu$, $\mathbf{z}_{\nu+1}$, *and* $\mathbf{z}_{\nu+2}$ *are linearly independent, then*

$$\zeta_{\nu+1} \leqslant \varphi^{[\psi]}(x_{\nu+1}). \tag{55}$$

Theorem 19 follows immediately from this fact and Corollary 2 (p. 457).

Indeed, we can assume that the determinant (48) is non-zero. Then the inequality (51) and Proposition 1 (see p. 442) lead to the estimate

$$1 \leqslant 6M_{\nu+2}\psi(M_{\nu+2})\psi(M_{\nu+1}) = 6x_{\nu+2}\psi(x_{\nu+2})\psi(x_{\nu+1}),$$

and (55) follows immediately.

To make the result of Theorem 19 more clear, we give a corollary about Diophantine exponents, also due to Jarník (see [13], part I of Theorem 1).

Corollary. *Let* $n \geqslant 2$ *and* $\dim_{\mathbb{Z}} \Theta \geqslant 3$, *and let* $\alpha(\Theta)$ *and* $\beta(\Theta)$ *be the suprema of those* γ *for which the respective conditions*

$$\limsup_{t\to+\infty} t^\gamma \psi_\Theta(t) < +\infty \tag{56}$$

and

$$\liminf_{t\to+\infty} t^\gamma \psi_\Theta(t) < +\infty, \tag{57}$$

hold for Jarník's function (20). *It is clear that* $1/n \leqslant \alpha(\Theta) \leqslant \beta(\Theta) \leqslant +\infty$. *If* $\alpha(\theta) < 1$, *then*

$$\beta(\Theta) \geqslant \frac{\alpha^2(\Theta)}{1-\alpha(\Theta)}. \tag{58}$$

Moreover, if $\alpha(\Theta) = 1$, *then* $\beta(\Theta) = +\infty$.

5.2. Case $m = 1$, $n = 3$. For $\alpha \in [1/3, 1]$ let

$$g_1(\alpha) = \frac{(1-\alpha)\alpha + \sqrt{(1-\alpha)^2\alpha^2 + 4\alpha(2\alpha^2 - 2\alpha + 1)}}{4\alpha^2 - 4\alpha + 2}.$$

Note that $g_1(\alpha)$ is a root of the equation

$$(2\alpha^2 - 2\alpha + 1)x^2 + \alpha(\alpha - 1)x - \alpha = 0.$$

It is easy to see that

$$g_1(\alpha) = \max_{\delta\geqslant 1,\, \gamma(1-\alpha)-\alpha>0} \min\left\{\delta, \frac{\alpha}{\gamma(1-\alpha)-\alpha}, \frac{1}{\alpha} - \left(\frac{1}{\alpha} - 1\right)\frac{\delta}{\gamma}\right\}, \tag{59}$$

$$g_1\left(\frac{1}{3}\right) = g_1(1) = 1.$$

Moreover, if $\alpha \in (1/3, 1)$, then $g_1(\alpha) > 1$.

Let $\alpha_0 \in (1/2, 1)$ be the root of the equation

$$x^3 - x^2 + 2x = 1.$$

Then for $1/3 < \alpha < \alpha_0$

$$g_1(\alpha) > \max\left\{1, \frac{\alpha}{1-\alpha}\right\}.$$

Theorem 20.[1] *Let $m = 1$ and $n = 3$. Suppose that the matrix Θ consists of numbers θ_1, θ_2, θ_3 linearly independent together with 1 over $\mathbb{Z}$. Then*

$$\beta(\Theta) \geqslant \alpha(\Theta) g_1(\alpha(\Theta)). \tag{60}$$

Obviously, the inequality (60) is stronger than the result of Theorem 19 (and its corollary) if $1/3 < \alpha(\Theta) < \alpha_0$.

Proof. Assume the inequality $\psi_\Theta(t) \leqslant \psi(t)$ for some continuous function $\psi(t)$ decreasing to zero. Suppose that the function $t \mapsto t\psi(t)$ increases to infinity as $t \to +\infty$.

We consider the sequence of best approximations $\mathbf{z}_\nu = (x_\nu, y_{1,\nu}, y_{2,\nu}, y_{3,\nu})$. From the linear independence condition it follows that there is an infinite sequence of pairs of indices $\nu < k$ such that $\nu \to +\infty$ and the following three conditions hold:

- both the triples
$$(\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1}), \qquad (\mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1})$$
consist of linearly independent vectors;
- there is a two-dimensional subspace π such that
$$\mathbf{z}_l \in \pi \quad \text{for } \nu \leqslant l \leqslant k, \quad \mathbf{z}_{\nu-1} \notin \pi, \quad \mathbf{z}_{k+1} \notin \pi;$$
- the four vectors $\mathbf{z}_{\nu-1}$, $\mathbf{z}_\nu$, $\mathbf{z}_k$, $\mathbf{z}_{k+1}$ are linearly independent.

Thus,

$$\begin{aligned} 1 &\leqslant \left| \det \begin{pmatrix} y_{1,\nu-1} & y_{2,\nu-1} & y_{3,\nu-1} & x_{\nu-1} \\ y_{1,\nu} & y_{2,\nu} & y_{3,\nu} & x_\nu \\ y_{1,k} & y_{2,k} & y_{3,k} & x_k \\ y_{1,k+1} & y_{2,k+1} & y_{3,k+1} & x_{k+1} \end{pmatrix} \right| \\ &\leqslant 24\zeta_{\nu-1}\zeta_\nu\zeta_k M_{k+1} \leqslant 24\psi(M_\nu)\psi(M_{\nu+1})\psi(M_{k+1})M_{k+1} \\ &< 24\psi(M_\nu)(\psi(M_{\nu+1}))^2 M_{k+1}. \end{aligned} \tag{61}$$

We must now consider three cases.

1°. For some $\gamma > 1$ there are infinitely many of the pairs (ν, k) under consideration such that

$$M_{k+1} \leqslant M_{\nu+1}^\gamma.$$

From the inequality (61) it follows that

$$\frac{1}{24\psi(M_\nu)} \leqslant M_{\nu+1}^\gamma \psi(M_{\nu+1})\psi(M_{\nu+1}^\gamma).$$

[1] Recently the author obtained a stronger result; see arXiv: 1009.0987.

Supposing in addition that the function $t \mapsto t^\gamma\psi(t)\psi(t^\gamma)$ is increasing, and denoting its inverse function by $\rho(t)$, we have

$$\zeta_\nu \leqslant \psi(M_{\nu+1}) \leqslant \psi\biggl(\rho\biggl(\frac{1}{24\psi(M_\nu)}\biggr)\biggr). \tag{62}$$

2°. For some $\delta \geqslant 1$ there are infinitely many of the pairs (ν, k) under consideration such that

$$M_{k+1} \geqslant M_k^\delta.$$

In this case we immediately get that

$$\zeta_k \leqslant \psi(M_{k+1}) \leqslant \psi(M_k^\delta). \tag{63}$$

3°. For infinitely many of the pairs of indices (ν, k) under consideration,

$$M_{\nu+1}^\gamma \leqslant M_{k+1} \leqslant M_k^\delta.$$

Then for the two-dimensional lattice $\Lambda = \mathbb{Z}^4 \cap \pi$ we have

$$\zeta_l M_{l+1} \asymp_\Theta \det\nolimits_2 \Lambda$$

for $\nu \leqslant l \leqslant k-1$. Therefore, we get the inequalities

$$\zeta_{k-1} \ll_\Theta \frac{M_{\nu+1}\psi(M_{\nu+1})}{M_k} \ll_\Theta M_k^{\delta/\gamma-1}\psi(M_k^{\delta/\gamma}) \leqslant M_{k-1}^{\delta/\gamma-1}\psi(M_{k-1}^{\delta/\gamma}) \tag{64}$$

(of course, it is necessary to make the additional assumption that the function $t^{\delta/\gamma-1}\psi(t^{\delta/\gamma})$ decreases monotonically).

To finish the proof of the theorem it is enough to consider a small $\varepsilon > 0$ and the function $\psi(t) = t^{-\alpha+\varepsilon}$. In view of (59), the inequality (60) follows from (62)–(64).

5.3. Case $m = 2$. We formulate Jarník's theorem about Diophantine type in [13]. This theorem is analogous to Theorem 19.

Theorem 21. *Let $m = 2$ and $n \geqslant 1$. Suppose that a non-degenerate matrix*

$$\Theta = \begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \dots & \dots \\ \theta_n^1 & \theta_n^2 \end{pmatrix}$$

is ψ-singular, with some function satisfying $\psi(t) = o(t^{-1})$ as $t \to +\infty$. Let

$$\varphi_2^{[\psi]}(t) = \psi\biggl(\frac{1}{6t\psi(t)}\biggr).$$

Then there exist infinitely many integer vectors $\mathbf{x} = (x_1, x_2)$ such that

$$\max_{1\leqslant j\leqslant n} \|\theta_j^1 x_1 + \theta_j^2 x_2\| \leqslant \varphi_2^{[\psi]}\Bigl(\max_{i=1,2}|x_i|\Bigr).$$

We note that $R(\Theta) \geqslant 3$ for a good matrix Θ under the conditions of the theorem. For $n = 1$ this follows from the statement (iii) of Theorem 7. For $n \geqslant 2$ it follows from the ψ-singularity of the matrix and from Theorem 8. To get a proof one must establish an analogue of Corollary 4 in § 4.1 (see p. 457). For some k there exist infinitely many ν such that the determinant

$$\det \begin{pmatrix} x_{1,\nu} & x_{2,\nu} & y_{k,\nu} \\ x_{1,\nu+1} & x_{2,\nu+1} & y_{k,\nu+1} \\ x_{1,\nu+2} & x_{2,\nu+2} & y_{k,\nu+2} \end{pmatrix}$$

is not equal to zero. (This statement needs some more arguments, which are omitted here.) Then for such values of ν we must establish an estimate similar to (51):

$$1 \leqslant 6M_{\nu+2}M_{\nu+1}\zeta_\nu \leqslant 6M_{\nu+2}M_{\nu+1}\psi(M_{\nu+1}).$$

To complete the proof, we use the monotonicity of the function ψ.

We do not give here the proof in detail. (For example, one must consider the case when the matrix Θ is not good.) All the details can be found in Jarník's original paper.

Jarník deduces the following corollary from Theorem 21.

Corollary. *Let $m = 2$ and $n \geqslant 1$, and suppose that the matrix Θ is non-degenerate. Consider the values $\alpha(\Theta)$ and $\beta(\Theta)$ defined as the suprema of those γ for which (56) and (57) hold, respectively. Then*

$$\beta(\Theta) \geqslant \alpha(\Theta)\big(\alpha(\Theta) - 1\big). \tag{65}$$

5.4. Case $m = n = 2$. For $\alpha \geqslant 1$ put

$$g_3(\alpha) = \frac{1 - \alpha + \sqrt{(1-\alpha)^2 + 4\alpha(2\alpha^2 - 2\alpha + 1)}}{2\alpha}.$$

This is a solution of the equation

$$\alpha x^2 + (\alpha - 1)x - (2\alpha^2 - 2\alpha + 1) = 0. \tag{66}$$

Note that $g_3(1) = 1$, and that $g_3(\alpha) > 1$ for $\alpha > 1$. Moreover, $1 \leqslant \alpha < \big((1+\sqrt{5}\,)/2\big)^2$ implies that

$$g_3(\alpha) > \max(1, \alpha - 1).$$

Theorem 22. *Let the numbers θ_j^i $(i, j = 1, 2)$ be linearly independent together with 1 over $\mathbb{Z}$. Consider the matrix*

$$\Theta = \begin{pmatrix} \theta_1^1 & \theta_1^2 \\ \theta_2^1 & \theta_2^2 \end{pmatrix}.$$

Then

$$\beta(\Theta) \geqslant \alpha(\Theta) g_3(\alpha(\Theta)).$$

Theorem 22 is stronger than Theorem 21 if $\alpha(\Theta) \in \big(1, \big((1+\sqrt{5}\,)/2\big)^2\big)$.

Proof. If $R(\Theta) = 2$, then from Theorem 8 it follows that $\alpha(\Theta) = 1$, and there is nothing to prove,

By Corollary 4 in § 2.1 (p. 446) it is not possible that $R(\Theta) = 3$. Thus, $R(\Theta) = 4$. Hence there is an infinite sequence of pairs of indices $\nu < k$ such that $\nu \to +\infty$ and the following three statements are valid:

- both the triples
$$(\mathbf{z}_{\nu-1}, \mathbf{z}_\nu, \mathbf{z}_{\nu+1}), \quad (\mathbf{z}_{k-1}, \mathbf{z}_k, \mathbf{z}_{k+1}),$$
consist of linearly independent vectors;
- there is a two-dimensional subspace π such that
$$\mathbf{z}_l \in \pi \quad \text{for } \nu \leqslant l \leqslant k, \qquad \mathbf{z}_{\nu-1} \notin \pi, \quad \mathbf{z}_{k+1} \notin \pi;$$
- the four vectors $\mathbf{z}_{\nu-1}$, $\mathbf{z}_\nu$, $\mathbf{z}_k$, $\mathbf{z}_{k+1}$ are linearly independent.

Now
$$1 \leqslant \left| \det \begin{pmatrix} x_{1,\nu-1} & x_{2,\nu-1} & y_{1,\nu-1} & y_{2,\nu-1} \\ x_{1,\nu} & x_{2,\nu} & y_{1,\nu} & y_{2,\nu} \\ x_{1,k} & x_{2,k} & y_{1,k} & y_{2,k} \\ x_{1,k+1} & x_{2,k+1} & y_{1,k+1} & y_{2,k+1} \end{pmatrix} \right| \leqslant 24\zeta_{\nu-1}\zeta_\nu M_k M_{k+1}.$$

Suppose that $\psi(t)$ is decreasing. Let $\psi_\Theta(t) \leqslant \psi(t)$. Then
$$1 \leqslant 24 M_{k+1} M_k \psi(M_{\nu+1})\psi(M_\nu). \tag{67}$$

We consider two cases.

1°. For some $\gamma > 1$ there are infinitely many of the pairs (ν, k) under consideration such that
$$M_{k+1} \geqslant M_k^\gamma.$$

Then we immediately get that
$$\zeta_k \leqslant \psi(M_{k+1}) \leqslant \psi(M_k^\gamma). \tag{68}$$

2°. For infinitely many of the pairs (ν, k) under consideration,
$$M_{k+1} \leqslant M_k^\gamma.$$

Then (67) implies that
$$M_k \geqslant \bigl(\psi(M_\nu)\bigr)^{-2/(1+\gamma)}. \tag{69}$$

We consider the two-dimensional lattice $\Lambda = \pi \cap \mathbb{Z}^4$ with fundamental volume $\det \Lambda$. The distance from a point $\mathbf{z} \in \pi$ to the two-dimensional subspace
$$\mathscr{L} = \{\mathbf{z} = (x_1, x_2, y_1, y_2) : \ \theta_1^1 x_1 + \theta_1^2 x_2 + y_1 = \theta_2^1 x_1 + \theta_2^2 x_2 + y_2 = 0\}$$
is proportional to the distance from $\mathbf{z}$ to the one-dimensional subspace $\mathscr{L} \cap \pi$. (The case $\mathscr{L} \cap \pi = \mathbf{0}$ can be treated similarly.) Let δ be the proportionality coefficient. The parallelepiped
$$\Bigl\{\mathbf{z} = (x_1, x_2, y_1, y_2) : \ |x| < M_{l+1}, \ \max_{1 \leqslant j \leqslant 3} |\theta_j x - y_j| < \zeta_l\Bigr\}$$

contains no non-zero integer points inside itself. Hence

$$\gamma_1(\Theta)\delta \det\Lambda \leqslant \zeta_l M_{l+1} \leqslant \gamma_2(\Theta)\delta\ \det\Lambda, \qquad \nu \leqslant l \leqslant k-1, \tag{70}$$

for some positive constants $\gamma_i(\Theta)$, $i = 1, 2$. From (69) and (70) we deduce that

$$\zeta_\nu \ll_\Theta \frac{\psi(M_k)M_k}{M_{\nu+1}} \ll_\Theta M_\nu^{-1}(\psi(M_\nu))^{-2/(1+\gamma)}\psi\big((\psi(M_\nu))^{-2/(1+\gamma)}\big). \tag{71}$$

To finish the proof we consider the function $\psi(t) = t^{-\alpha(\Theta)+\varepsilon}$ with a small positive number ε. Since $\gamma = g_3(\alpha(\Theta))$ satisfies (66), we obtain the statement of Theorem 22.

5.5. Case $m > 2$. In the case $m > 2$ Jarník in [13] uses more complicated arguments. That is why he proves only a statement concerning Diophantine exponents but does not considers a result with a function ψ of general type.

Theorem 23. *Let $m \geqslant 3$. Suppose that for a non-degenerate matrix Θ*

$$\alpha(\Theta) > (5m^2)^{m-1}.$$

Then

$$\beta(\Theta) \geqslant (\alpha(\Theta))^{m/(m-1)} - 3\alpha(\Theta).$$

We do not give the proof of this theorem. It is based on an elegant construction of a sequence of *linearly independent* best approximation vectors. The author is sure that the result of Theorem 23 is not optimal and can be improved. In the next subsection we consider such an improvement in a particular case.

5.6. Case $m = 3$, $n = 1$. For $\alpha \geqslant 3$ we define the functions

$$g_2(\alpha) = \sqrt{\alpha + \frac{1}{\alpha^2} - \frac{7}{4}} + \frac{1}{\alpha} - \frac{1}{2}, \qquad h(\alpha) = \alpha - g_2(\alpha) - 1.$$

We note that $g_2(\alpha)$ and $h(\alpha)$ increase to infinity as $\alpha \to +\infty$ and that

$$g_2(3) = h(3) = 1, \qquad g_2(\alpha) \leqslant \alpha - 2.$$

Theorem 24. *Consider a row matrix $\Theta = (\theta^1, \theta^2, \theta^3)$ consisting of numbers linearly independent together with 1 over $\mathbb{Z}$. Then the exponents $\alpha(\Theta)$ and $\beta(\Theta)$ are connected by the relation*

$$\beta(\Theta) \geqslant \alpha(\Theta) g_2(\alpha(\Theta)). \tag{72}$$

Proof. We consider the sequence of best approximation vectors $\mathbf{z}_\nu$.

First of all, consider the case $R(\Theta) = 3$. Then for sufficiently large ν the vectors $\mathbf{z}_\nu$ lie in some three-dimensional completely rational subspace π, and we are actually dealing with the best approximations of the *two-dimensional* subspace $\pi \cap \mathscr{L}(\Theta)$ by points of the three-dimensional lattice $\mathbb{Z}^4 \cap \pi$ (the definition of the subspace $\mathscr{L}(\Theta)$ is given in § 2.1). Then we can apply Theorem 21, and its corollary gives the estimate (65), which is better than (72).

Let us now consider the case $R(\Theta) = 4$. In this case there are infinitely many pairs of indices $\nu < k$ $(\nu \to +\infty)$ such that:

- the two triples

$$\mathbf{z}_{\nu-1}, \mathbf{z}_{\nu}, \mathbf{z}_{\nu+1}, \qquad \mathbf{z}_{k-1}, \mathbf{z}_{k}, \mathbf{z}_{k+1}$$

consist of linearly independent vectors;
- there is a two-dimensional completely rational subspace π such that

$$\mathbf{z}_l \in \pi \quad \text{for } \nu \leqslant l \leqslant k, \qquad \mathbf{z}_{\nu-1} \notin \pi, \quad \mathbf{z}_{k+1} \notin \pi;$$

- the four vectors $\mathbf{z}_{\nu-1}$, $\mathbf{z}_{\nu}$, $\mathbf{z}_{\nu+1}$, $\mathbf{z}_{k+1}$ are linearly independent.

If Θ is ψ-singular and the functions $\psi(t)$ and $t\psi(t)$ are monotone, then

$$\begin{aligned}1 \leqslant \left|\det\begin{pmatrix} x_{1,\nu-1} & x_{2,\nu-1} & x_{3,\nu-1} & y_{\nu-1} \\ x_{1,\nu} & x_{2,\nu} & x_{3,\nu} & y_{\nu} \\ x_{1,\nu+1} & x_{2,\nu+1} & x_{3,\nu+1} & y_{\nu+1} \\ x_{1,k+1} & x_{2,k+1} & x_{3,k+1} & y_{k+1}\end{pmatrix}\right| \\ \leqslant 24\zeta_{\nu-1}M_\nu M_{\nu+1}M_{k+1} \leqslant 24\psi(M_\nu)M_\nu M_{\nu+1}M_{k+1}. \end{aligned} \tag{73}$$

We must consider three cases.

1°. There are infinitely many pairs (ν, k) under consideration such that

$$M_{k+1} \leqslant M_\nu^{h(\alpha(\Theta))}.$$

Then from (73) we deduce that

$$M_{\nu+1} \geqslant \frac{1}{24\psi(M_\nu)M_\nu^{1+h(\alpha(\Theta))}}, \qquad \zeta_\nu \leqslant \psi\biggl(\frac{1}{24\psi(M_\nu)M_\nu^{1+h(\alpha(\Theta))}}\biggr),$$

and the last inequality immediately implies (72).

2°. For infinitely many pairs (ν, k) under consideration,

$$M_{k+1} \geqslant M_k^{g_2(\alpha(\Theta))}.$$

Then we immediately get that

$$\zeta_k \leqslant \psi(M_{k+1}) \leqslant \psi\bigl(M_k^{g_2(\alpha(\Theta))}\bigr),$$

and (72) follows.

3°. There are infinitely many pairs (ν, k) under consideration such that

$$M_\nu^{h(\alpha(\Theta))} \leqslant M_{k+1} \leqslant M_k^{g_2(\alpha(\Theta))}.$$

In this case a detailed investigation of the best approximations in the subspace π is necessary. Consider a projection of π on the subspace $\mathscr{L}(\Theta)$. We can assume this projection to be some two-dimensional subspace π^* intersecting π in a line $\ell = \pi \cap \pi^*$. For a point $\mathbf{z} \in \pi$ the distance from $\mathbf{z}$ to the subspace $\mathscr{L}(\Theta)$ is proportional to the distance from $\mathbf{z}$ to the one-dimensional subspace ℓ. Let δ be the coefficient of this proportionality. The vectors $\mathbf{z}_l$ are best approximation vectors, so they, regarded as vectors in the lattice $\Lambda = \mathbb{Z}^4 \cap \pi$, are automatically best approximations of ℓ by points of Λ with respect to the induced norm. Let

$\det\Lambda$ be the two-dimensional fundamental volume of the lattice Λ. It is clear that for some positive constants $\gamma_i(\Theta)$ $(i=1,2)$

$$\gamma_1(\Theta)\delta\det\Lambda \leqslant \zeta_l M_{l+1} \leqslant \gamma_2(\Theta)\delta\det\Lambda, \qquad \nu\leqslant l\leqslant k-1.$$

In particular,

$$\zeta_\nu M_{\nu+1} \leqslant \frac{\gamma_2(\Theta)}{\gamma_1(\Theta)}\,\zeta_{k-1}M_k.$$

We take into account that $\zeta_{k-1}\leqslant\psi(M_k)$. From the condition which defines this case we deduce that

$$\zeta_\nu \leqslant \frac{\gamma_2(\Theta)}{\gamma_1(\Theta)}\,\frac{\psi(M_k)M_k}{M_{\nu+1}} \leqslant \frac{\gamma_2(\Theta)}{\gamma_1(\Theta)}\,\psi\bigl(M_\nu^{h(\alpha(\Theta))/g_2(\alpha(\Theta))}\bigr)M_\nu^{h(\alpha(\Theta))/g_2(\alpha(\Theta))-1}.$$

Since

$$\alpha(g_2(\alpha))^2+(\alpha-2)g_2(\alpha)-(\alpha-1)^2=0,$$

we again have (72).

The theorem is proved.

6. Inhomogeneous approximations

6.1. One-dimensional setting. Given real numbers θ and α, we consider the quantity

$$\lambda(\theta,\alpha)=\liminf_{x\to\infty}|x|\cdot\|x\theta-\alpha\|$$

(here we suppose that x takes integer values). By the classical Minkowski theorem (see [23], Chap. III, Theorem II), for any θ and any α not of the form $a\theta+b$ with integers a and b we have the inequality

$$\lambda(\theta,\alpha)\leqslant\frac{1}{4}\,.$$

Let

$$\lambda(\theta)=\lambda(\theta,0) \quad\text{and}\quad \mu(\theta)=\sup_\alpha\lambda(\theta,\alpha),$$

where the supremum is taken over all α not of the form $a\theta+b$ with integers a, b.

Khintchine in [5] proved the inequality

$$\mu(\theta)\leqslant\frac{\sqrt{1-4\lambda^2(\theta)}}{4}\,, \tag{74}$$

which is sharp for certain values of θ. For example, for θ equivalent to a purely periodic continued fraction $[0;k,k,k,\dots]$, where the partial quotient k is equal to 1 or is even, the inequality (74) turns into an equality. It follows from (74) that the equality $\mu(\theta)=1/4$ is possible only for θ with unbounded partial quotients in its continued fraction expansion.

There are some other results in Khintchine's paper [5]. Various papers have been devoted to the study of values of the function $\lambda(\theta,\alpha)$ as well as of some other related functions, such as the 'one-sided' function

$$k(\theta,\alpha)=\liminf_{x\to+\infty} x\|x\theta-\alpha\|.$$

Here we can mention papers by Cassels [45], Barnes [46], and Cusick and Pollington [47].

We now state a simple (but nevertheless non-trivial) one-dimensional result which was proved in Khintchine's paper [1] (Satz 4).

Theorem 25. *There exists an absolute positive constant γ with the property that for any real number θ there is a real number α such that for all positive integers x*

$$\|x\theta-\alpha\|\geqslant\frac{\gamma}{x}\,.$$

In Cassels' book [23] the result of Theorem 25 is proved with $\gamma=1/51$ (see [23], Chap. V, Theorem XI). In the Russian translation there is a strange remark after Chap. V concerning possible bounds for the exact value of γ. The best result is probably due to Godwin [48].

Below we discuss multidimensional generalizations of Theorem 25 along with the steps of its proof, since the proof of this theorem was a starting point for all the multidimensional results, including Jarník's theorems, which will be discussed in the next subsection.

6.2. Multidimensional theorems. A certain connection between homogeneous and inhomogeneous linear Diophantine approximations appears in the proof of Theorem 25. The study of this connection was extended to the multidimensional situation. It turned out that the singularity characteristics of the corresponding homogeneous systems are of importance. Khintchine conducted repeated studies of inhomogeneous approximations. In 1936 he proved in [4] a fundamental multidimensional result, stated below (more precisely, he stated this result for arbitrary values of m but proved it only for $m=2$).

Theorem 26 (Khintchine [4]). *The following two conditions (i) and (ii) are equivalent*:

(i) *the set of real numbers $\theta_1,\dots,\theta_m$ satisfies the condition*

$$\limsup_{t\to+\infty} t^m \min_{\mathbf{x}\in\mathbb{Z}^m:\,0<M(\mathbf{x})\leqslant t}\biggl\|\sum_{1\leqslant i\leqslant m}\theta_i x_i\biggr\|>0;$$

(ii) *the set of real numbers $\theta_1,\dots,\theta_m$ satisfies for any real number α the condition*

$$\liminf_{t\to+\infty} t^m \min_{\mathbf{x}\in\mathbb{Z}^m:\,M(\mathbf{x})\leqslant t}\biggl\|\sum_{1\leqslant i\leqslant m}\theta_i x_i-\alpha\biggr\|<+\infty.$$

Obviously, the condition (i) is equivalent to the regularity of the system Θ in (1) with the given m and $n=1$.

We note that the proofs of all the results in the present subsection rely on ideas in Khintchine's papers [1], [4].

In 1948 Khintchine proved the following result in [7].

We recall his notation. A system Θ is called a *Tchebyshev system* if for any tuple $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ there exists a constant $\Gamma = \Gamma(\alpha)$ such that the system of Diophantine inequalities

$$\max_{1 \leqslant j \leqslant n} \|L_j(\mathbf{x}) - \alpha_j\| \leqslant \Gamma\Big(\max_{1 \leqslant i \leqslant m} |x_i|\Big)^{-m/n}$$

has integer solutions with arbitrarily large values of $\max_{1 \leqslant i \leqslant m} |x_i|$.

The main result in [7] is as follows.

Theorem 27. *A system Θ is regular if and only if it is a Tchebyshev system.*

Thus, Theorem 26 above is a special case of Theorem 27 ($n = 1$).

One can find a proof of this theorem not only in Khintchine's original paper [7] but also in the book [23], Chap. X.

We should note that Jarník obtained more general results earlier (in the papers [10] and [11] in 1939 and 1941, respectively; see also [12]). We state these below.

Together with the 'homogeneous' Jarník function (20) it is convenient to consider an 'inhomogeneous' function

$$\psi_{\Theta,\alpha}(t) = \min_{\mathbf{x} \in \mathbb{R}^m \colon M(\mathbf{x}) \leqslant t} \ \max_{1 \leqslant j \leqslant n} \|L_j(\mathbf{x}) - \alpha_j\|, \qquad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n,$$

and the function

$$\Psi_{\Theta}^{[\mathrm{inhom}]}(t) = \sup_{\alpha \in [0,1]^n} \psi_{\Theta,\alpha}(t).$$

One can take the minimum over all *non-zero* vectors $\mathbf{x}$ in the definition of the 'inhomogeneous' Jarník function. The final result will be the same.

It is clear that the Jarník function (20) satisfies the equality

$$\psi_{\Theta}(t) = \psi_{\Theta,\mathbf{0}}(t).$$

Together with the system Θ we consider the transposed system ${}^t\Theta$. For the Jarník function $\psi_{{}^t\Theta}(t)$ we obviously have

$$\psi_{{}^t\Theta}(t) = \min_{\mathbf{x} \in \mathbb{R}^n \colon M(x) < t} \ \max_{1 \leqslant i \leqslant m} \|L_i^*(\mathbf{x})\|,$$

where

$$L_i^*(\mathbf{x}) = \sum_{j=1}^{n} \theta_j^i x_j.$$

Everywhere in this subsection we suppose that the function $\psi(t)$ decreases to zero. Let $\rho(t)$ be the function inverse to the function $1/\psi(t)$.

First of all, we formulate a simpler result contained in [10] (Theorem 1 in [10]).

Theorem 28 (Jarník [10]). *Suppose that $\psi(t)$ has the property that for some $\eta > 0$ the function $1/(t^\eta \psi(t))$ increases to infinity as $t \to +\infty$. Assume that for sufficiently large t*

$$\psi_{{}^t\Theta}(t) > \psi(t). \tag{75}$$

Then for sufficiently large t

$$\Psi_\Theta^{[\text{inhom}]}(t) \leqslant \frac{((m+n)!\,(m+n))^{(\eta+1)/\eta}}{\rho(t)}\,. \tag{76}$$

In the case $\psi_{{}^t\Theta}(t) = ct^{-n/m}$ with a positive constant c, we obtain from Theorem 28 the following statement.

Corollary. *Suppose that the transposed system ${}^t\Theta$ is regular. Then Θ is a Tchebyshev system.*

The following theorem is actually a result of Jarník in the paper [11] (Theorem 7 in [11]; see also [12]). The difference between Jarník's result and the theorem below is that in Theorem 29 we have explicit constants.

Theorem 29. *Suppose that for all sufficiently large t*

$$\psi_{{}^t\Theta}(t) \leqslant \psi(t). \tag{77}$$

Then there exists a tuple $(\alpha_1, \dots, \alpha_n)$ of real numbers such that for sufficiently large t

$$\psi_{\Theta,\alpha}(t) \geqslant \frac{1}{24 n^{3/2} \rho(8mt)}\,. \tag{78}$$

Such a statement is the main complication in the problem under consideration. It is related to Khintchine's fundamental Theorem 25 which was proved in [1]. The proof of this theorem will be discussed in the next subsection. Here we mention two important corollaries.

If we suppose that $\psi_{{}^t\Theta}(t) = o(t^{-n/m})$ for $t \to +\infty$, then we obtain the following statement.

Corollary. *If the transposed system ${}^t\Theta$ is singular, then the system Θ is not a Tchebyshev system.*

This corollary combined with the corollary of Theorem 28 leads to the following statement, which it is natural to attribute to Jarník.

Theorem 30. *The transposed system ${}^t\Theta$ is regular if and only if it is a Tchebyshev system.*

To see that Khintchine's Theorem 27 is equivalent to Jarník's Theorem 30, we recall a transference theorem proved by Khintchine in 1948 in [6] (see also the book [23], Chap. V, Theorem XII).

Theorem 31. *A system Θ is singular if and only if the transposed system ${}^t\Theta$ is singular.*

But Jarník's paper [11] (1941) contains the following statement.

Theorem 32. *If there is a positive constant γ_1 such that for sufficiently large t*

$$\psi_\Theta(t) \geqslant \gamma_1 t^{-m/n},$$

then there is a positive constant γ_2 such that for sufficiently large t

$$\psi_{{}^t\Theta}(t) \geqslant \gamma_2 t^{-n/m}.$$

Theorems 31 and 32 are obviously equivalent!

We discuss some other transference theorems in § 8.

The proofs of Jarník's Theorems 28 and 29 do not differ radically from the proof of Theorem 31 given by Khintchine. Khintchine proved that a regular system Θ is a Tchebyshev system. To do this it is sufficient to apply Minkowski's theorem to the system of $m+n$ linear forms

$$L_j(\mathbf{x}) - y_j - \xi_j u, \qquad j = 1, \dots, n, \quad x_j, \quad j = 1, \dots, m,$$

in the $m+n+1$ variables $x_1, \dots, x_n, y_1, \dots, y_m, u$ (here the ξ_j are some new real parameters). Then by certain constructive arguments (to be discussed in § 6.3) and by transference arguments he proves the converse statement. Jarník uses transference arguments in Theorem 28, while Theorem 29 is a constructive theorem.

We would like to discuss one more corollary of Theorem 29. By the Minkowski convex body theorem (mentioned at the beginning of § 1), if we let $\psi(t) = t^{-n/m}$, then the condition (77) in Jarník's Theorem 29 becomes an empty condition. Hence, we deduce the following result as a corollary (this result was stated in Cassels' book [23] as Theorem X in Chap. V).

Theorem 33. *For any positive integers n and m there is a positive constant $\Gamma_{m,n}$ with the property that for any matrix Θ there exists a vector $(\alpha_1, \dots, \alpha_n)$ such that*

$$\inf_{\mathbf{x}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} \Bigl(\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x}) - \alpha_j\|\Bigr)^n \Bigl(\max_{1\leqslant i\leqslant m} |x_i|\Bigr)^m > \Gamma_{m,n}.$$

This theorem is a direct generalization of Theorem 25, proved by Khintchine in [1], and there is an interesting story related to it which we discuss it in § 6.4.

In concluding the present subsection we note that in Jarník's wonderful paper [11] there are some other results which we do not give here. In particular, there is a certain metric result.

6.3. On the proof of Theorem 29. The proof of Theorem 33 in the book [23] can be transformed into a proof of Theorem 29. The original arguments by Jarník are more complicated. Here we give the scheme of a proof of Theorem 29, following [23]. We restrict ourselves to the situation when the numbers θ^i_j are linearly independent together with 1 over $\mathbb{Z}$. This is done for simplicity.

1. Consider the sequence of best approximations

$$\mathbf{w}_\nu = (u_{1,\nu}, \dots, u_{n,\nu}, v_{1,\nu}, \dots, v_{m,\nu})$$

for the transposed system ${}^t\Theta$. From this sequence we must pick a subsequence $\mathbf{w}_{\nu_k}$ in such a way that

$$\begin{gathered} \max_{1\leqslant i\leqslant m} \|L^*_i(\mathbf{u}_{\nu_k})\| = \psi_{{}^t\Theta}\biggl(\frac{M(\mathbf{u}_{\nu_{k+1}})}{3\sqrt{n}}\biggr), \\ M(\mathbf{u}_{\nu+1}) \geqslant 3\sqrt{n}\, M(\mathbf{u}_\nu), \qquad M(\mathbf{u}_\nu) = \max_{1\leqslant j\leqslant n} |u_{j,\nu+1}| \end{gathered} \tag{79}$$

(see Lemma 4 in § 6 of Chap. V in [23], or the arguments in § 2, 2 in [7], or the construction of Lemma 1 in [11]).

2. We need the following result on Diophantine approximation with a lacunary sequence of vectors, taken from [23] (Lemma 2 in § 6 of Chap. V). It generalizes a lemma of Khintchine (Hilfssatz 3, [1]). We discuss a further history of this statement in § 13.1 in the Appendix.

Lemma 2. *Suppose that a sequence of vectors* $\mathbf{u}_k = (u_{1,k}, \dots, u_{n,k}) \in \mathbb{R}^n$, *satisfies the condition*

$$\max_{1\leqslant j\leqslant n} |u_{j,k+1}| \geqslant 3\sqrt{n} \max_{1\leqslant j\leqslant n} |u_{j,k}|.$$

Then there exist real numbers $\alpha_1, \dots, \alpha_n$ *such that*

$$\Bigl\| \sum_{1\leqslant j\leqslant n} u_{j,k}\alpha_j \Bigr\| \geqslant \frac{1}{4}, \qquad k = 1, 2, 3, \dots .$$

The additional factor $\sqrt{n}$ (in comparison with the statement in [23]) appears because we use the sup-norm and not the Euclidean norm.

3. Then one must prove that the numbers α_j are precisely the inhomogeneities whose existence is asserted in Theorem 29. For this purpose we use the identity

$$\begin{aligned}\sum_{1\leqslant j\leqslant n} u_j\alpha_j &= \sum_{1\leqslant j\leqslant n} u_j(\alpha_j - L_j(\mathbf{x})) + \sum_{1\leqslant j\leqslant n} u_j L_j(\mathbf{x})\\ &= \sum_{1\leqslant j\leqslant n} u_j(\alpha_j - L_j(\mathbf{x})) + \sum_{1\leqslant i\leqslant m} x_i L_i^*(\mathbf{u}),\end{aligned}$$

where

$$\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m, \qquad \mathbf{u} = (u_1, \dots, u_n) \in \mathbb{R}^n.$$

We apply this identity to the vector $\mathbf{u}_{\nu_k}$ defined in item 1 of the proof (see (79)), for the numbers α_j in Lemma 2 and for an arbitrary vector $\mathbf{x} \in \mathbb{R}^m$. Taking absolute values and using Lemma 2, we get that

$$\frac{1}{4} \leqslant nM(\mathbf{u}_{\nu_k}) \max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x}) - \alpha_j\| + m \max_{1\leqslant i\leqslant m} |x_i| \cdot \psi_{t\Theta}\Bigl(\frac{M(\mathbf{u}_{\nu_{k+1}})}{3\sqrt{n}}\Bigr).$$

For a vector $\mathbf{x}$ we now choose k from the condition

$$\psi_{t\Theta}\Bigl(\frac{M(\mathbf{u}_{\nu_k})}{3\sqrt{n}}\Bigr) \geqslant \frac{1}{8m \max_{1\leqslant i\leqslant m} |x_i|} \geqslant \psi_{t\Theta}\Bigl(\frac{M(\mathbf{u}_{\nu_{k+1}})}{3\sqrt{n}}\Bigr), \tag{80}$$

whence

$$\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x}) - \alpha_j\| \geqslant \frac{1}{8nM(\mathbf{u}_{\nu_k})}. \tag{81}$$

But from the condition (77) of our theorem and from (80) we obtain

$$\psi\Bigl(\frac{M(\mathbf{u}_{\nu_k})}{3\sqrt{n}}\Bigr) \geqslant \psi_{t\Theta}\Bigl(\frac{M(\mathbf{u}_{\nu_k})}{3\sqrt{n}}\Bigr) \geqslant \frac{1}{8m \max_{1\leqslant i\leqslant m} |x_i|}.$$

By the definition of the function $\rho(\cdot)$ we now get that

$$\rho\Bigl(8m \max_{1\leqslant i\leqslant m} |x_i|\Bigr) \geqslant \frac{M(\mathbf{u}_{\nu_k})}{3\sqrt{n}}.$$

We substitute the last inequality into (81) to obtain (78) with $t = \max_{1\leqslant i\leqslant m} |x_i|$.

6.4. On Theorem 33. At present we see increasing interest in certain problems related to linear inhomogeneous Diophantine approximation. In particular, Theorem 33 has been generalized by several authors. In this subsection we discuss the history and give the scheme of a proof of the strongest result so far.

Kleinbock proved the following theorem [49].

Theorem 34. *Let $\mathscr{B}$ denote the set of real $(m+1)\times n$ matrices of the form*

$$(\theta_j^i,\eta_j),\qquad 1\leqslant i\leqslant m,\quad 1\leqslant j\leqslant n,$$

such that

$$\inf_{\mathbf{x}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}}\Bigl(\max_{1\leqslant j\leqslant n}\|L_j(\mathbf{x})-\eta_j\|\Bigr)^n\Bigl(\max_{1\leqslant i\leqslant m}|x_i|\Bigr)^m>0.$$

Then $\mathscr{B}$ is a set of full Hausdorff dimension in $\mathbb{R}^{mn+n}$.

Kleinbock's proof relies on a consideration of certain flows on homogeneous spaces. A simplified proof was given by Bugeaud, Harrap, Kristensen, and Velani in [50]. Moreover, they obtained the following stronger result.

Theorem 35. *For a collection*

$$\Theta=\{\theta_j^i,\ 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\}$$

of real numbers consider the set $\mathscr{B}(\Theta)$ of vectors $(\eta_1,\dots,\eta_n)$ such that

$$\inf_{\mathbf{x}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}}\Bigl(\max_{1\leqslant j\leqslant n}\|L_j(\mathbf{x})-\eta_j\|\Bigr)^n\Bigl(\max_{1\leqslant i\leqslant m}|x_i|\Bigr)^m>0.$$

Then $\mathscr{B}(\Theta)$ is a set of full Hausdorff dimension in $\mathbb{R}^n$.

To formulate further improvements, we need to use the theory of (α,β)-games constructed by Schmidt. The main concepts and results from this theory are collected in § 13.4 in the Appendix; they are necessary for understanding the results below in this subsection.

We now state a recent result of Tseng in [51].

Theorem 36. *For any real number θ consider the set $\mathscr{B}$ of real numbers η such that*

$$\inf_{x\in\mathbb{Z}\setminus\{0\}}|x|\cdot\|\theta x+\eta\|>0.$$

Then this set is an α-winning set for every $\alpha\in(0,1/8)$.

Remark 2.3 in [51] asserts that Tseng and Einsiedler obtained a generalization of Theorem 36 to the case of a system of linear forms of arbitrary dimension, and here we must refer to the very recent preprint [52].

In [53] the author modified the proof of Theorem 33 and obtained the following result.

Theorem 37. *Let $\alpha \in (0,1/2]$. For a collection*

$$\Theta = \{\theta_j^i,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$$

of real numbers consider the set $\mathscr{B}(\Theta)$ of vectors $(\eta_1,\dots,\eta_n)$ such that

$$\inf_{\mathbf{x}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} \Bigl(\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})-\eta_j\|\Bigr)^n \Bigl(\max_{1\leqslant i\leqslant m} |x_i|\Bigr)^m > 0,$$

Then this is an α-winning set in $\mathbb{R}^n$.

A more general result is valid.

Theorem 38. *Let $\alpha \in (0,1/2]$ and suppose that the function $\psi(t)$ strictly decreases to zero as $t \to +\infty$. Let $\rho(t)$ be the function inverse to the function $1/\psi(t)$. Suppose that for any $w \geqslant 1$*

$$\sup_{x\geqslant 1} \frac{\rho(wx)}{\rho(x)} < +\infty.$$

For a matrix Θ consider the Jarník function $\psi_{{}^t\Theta}$ and suppose that $\psi_{{}^t\Theta}(t) \leqslant \psi(t)$ for sufficiently large t. Then the set $\mathscr{B}(\Theta)$ of all vectors $(\eta_1,\dots,\eta_n)$ such that

$$\inf_{\mathbf{x}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} \Bigl(\max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})-\eta_j\|\Bigr)\rho\Bigl(\max_{1\leqslant i\leqslant m} |x_i|\Bigr) > 0,$$

is an α-winning set in $\mathbb{R}^n$.

It is easy to see (since $\psi_{{}^t\Theta}(t) \leqslant t^{-n/m}$ always holds) that Theorem 37 is a particular case of Theorem 38.

We give the scheme of the proof of this theorem below. To get a proof of Theorem 38, we only need a certain modification of Lemma 2 on a lacunary sequence of vectors in § 6.3.

Lemma 3. *Consider a sequence $\Lambda \subset \mathbb{Z}^n$ of integer vectors $\mathbf{u}^{(r)} = \bigl(u_1^{(r)},\dots,u_n^{(r)}\bigr) \in \mathbb{Z}^n$ such that the sequence of their lengths*

$$t_r = \Bigl(\bigl(u_1^{(r)}\bigr)^2 + \dots + \bigl(u_n^{(r)}\bigr)^2\Bigr)^{1/2} \tag{82}$$

satisfies the lacunarity condition

$$\frac{t_{r+1}}{t_r} \geqslant M, \qquad r = 1,2,3,\dots, \tag{83}$$

for some $M > 1$. Then the set

$$N(\Lambda) = \{\eta = (\eta_1,\dots,\eta_n) \in \mathbb{R}^n : \text{ there exists a } c(\eta) > 0 \text{ such that } \|u_1^{(r)}\eta_1 + \dots + u_n^{(r)}\eta_n\| \geqslant c(\eta) \text{ for all } r \in \mathbb{N}\}$$

is α-winning for all $\alpha \in (0,1/2]$.

Under the conditions of Theorem 38 a vector $\eta \in N(\Lambda)$ (for a reasonable choice of $\mathbf{u}^{(r)}$) will belong to the set $\mathscr{B}(\Theta)$. This follows from the classical arguments in the proof of Theorem 29. We recalled these arguments in § 6.3. The proofs of the particular case (Theorem 37) and the general case (Theorem 38) are identical.

Here we give a sketch of the proof of Lemma 3.

As usual, for $\alpha, \beta \in (0,1)$ let $\gamma = 1 + \alpha\beta - 2\alpha > 0$.

Consider a ball (in the Euclidean norm, which is more convenient here) $B \subset \mathbb{R}^n$ with centre O and radius ρ. We denote by $S = \partial B$ its boundary and by μ the normalized Lebesgue measure on S $\bigl(\text{so } \int_S d\mu = \mu S = 1\bigr)$.

Let $x \in S$, and let $\pi(x) \subset \mathbb{R}^n$ be the $(n-1)$-dimensional affine subspace passing through O and orthogonal to the one-dimensional subspace passing through O and x. Let $\Pi(x)$ be the half-space with boundary $\pi(x)$ and such that $x \in \Pi(x)$.

For the given $\alpha, \beta \in (0,1)$ consider the half-space $\Pi_{\alpha,\beta,\rho}(x)$ such that $\Pi_{\alpha,\beta,\rho}(x) \subset \Pi(x)$ and the distance from $\Pi_{\alpha,\beta,\rho}(x)$ to O is equal to $\gamma\rho/2$. Let

$$\Omega(x) = S \cap \Pi_{\alpha,\beta,\rho}(x), \qquad \Omega^*(x) = \bigcup_{y \in S\colon \Pi(y) \supset \Omega(x)} \{y\}.$$

It is clear that the measure $\mu\Omega^*(x)$ does not depend on $x \in S$. Let

$$\omega = \omega(\alpha, \beta) = \mu(\Omega^*(x)) \in (0,1). \tag{84}$$

By means of a certain mean value argument we obtain the following statement.

Lemma 4. *Take arbitrary affine subspaces $\pi_1, \dots, \pi_k$ of dimension $n-1$. Then there exists a point $x \in S$ such that*

$$\Omega(x) \cap \pi_j = \varnothing$$

for at least $\lceil \omega k \rceil$ values of the index j.

The following lemma is due to Schmidt ([24], Chap. 3, Lemma 1B).

Lemma 5. *Suppose that t satisfies*

$$(\alpha\beta)^t < \frac{\gamma}{2}\,.$$

Assume that in the game a black ball B_j occurs. Let π be an $(n-1)$-dimensional affine subspace π passing through the centre of B_j. Then White can play in such a way that the black ball B_{j+t} belongs to the half-space $\Pi_{\alpha,\beta,\rho_j}(x)$, whose boundary is parallel to the subspace π.

Taking the parameters

$$t = t(\alpha, \beta) = \left\lceil \frac{\log(\gamma/2)}{\log(\alpha\beta)} \right\rceil, \qquad \tau_k = t \left\lceil \frac{\log k}{\log(1/(1-\omega))} \right\rceil \tag{85}$$

(ω is defined in (84)), we get the following conclusion.

Corollary. *Suppose that a ball B_j with radius ρ_j occurs as a black ball, and let π_i $(1 \leqslant i \leqslant k)$ be a collection of affine subspaces. Then White can play in such a way that for any point $x \in B_{j+\tau_k}$ the distance from x to any of the subspaces π_i $(1 \leqslant i \leqslant k)$ is greater than $(\rho_{j+\tau_k}\gamma)/2$.*

We now take the parameter $k = k(\alpha,\beta,M)$ such that

$$\tau_k \frac{\log(1/(\alpha\beta))}{\log M} + 2 < k. \tag{86}$$

Without loss of generality we can suppose in addition to the lacunarity condition (83) that $t_{r+1}/t_r \leqslant M^2$ for $r = 1,2,3,\dots$. The choice of parameters (85), (86) enables White to play in such a way that in τ_k steps of the game it is possible to 'escape' from all the $q = r_{j-1} - r_j < k$ families of 'dangerous' subspaces

$$\{\mathbf{y} = (y_1,\dots,y_n) \in \mathbb{R}^n : \ u_1^{(r)} y_1 + \dots + u_n^{(r)} y_n = a\}, \qquad a \in \mathbb{Z}, \quad r_j \leqslant r < r_{j+1}.$$

7. Spaces of lattices

Problems in Diophantine approximation are related naturally to problems of the behavior of collections (orbits) of certain lattices. Many papers have been devoted to the topic (see, for example, the papers [49], [54]–[57] by Margulis, Kleinbock, and Weiss, the survey [58] by Gorodnik, and the bibliographies in these papers). Below we consider two problems related to singular matrices.

7.1. The Davenport–Schmidt metrical theorem. As noted at the end of § 1.2, singular matrices Θ form in nm-dimensional space a set of zero Lebesgue measure. This can be said in other words as follows. Consider the set $\mathfrak{T}_\mu \subset \mathbb{R}^{mn}$ of those Θ such that for sufficiently large t any domain of the form

$$\biggl\{\mathbf{x} \in \mathbb{R}^n : \max_{1\leqslant j\leqslant n} \|L_j(\mathbf{x})\| \leqslant \frac{\mu}{t}, \ 0 < \max_{1\leqslant i\leqslant m} |x_i| \leqslant \mu t^{n/m}\biggr\} \tag{87}$$

contains a non-zero integer point. It is clear that $\mathfrak{T}_{\mu_1} \subset \mathfrak{T}_{\mu_2}$ if $\mu_1 < \mu_2$. Then the intersection

$$\bigcap_{\mu>0} \mathfrak{T}_\mu$$

has zero Lebesgue measure.

A wonderful improvement of this result was obtained by Davenport and Schmidt.

Theorem 39 (Davenport and Schmidt [39], [25]). *For any μ with $0 < \mu < 1$ the set $\mathfrak{T}_\mu$ has zero Lebesgue measure.*

Note that $\mathfrak{T}_1 = \mathbb{R}^{mn}$ for $\mu = 1$, by Minkowski's convex body theorem (cited at the beginning of the paper).

Remark. The only case considered in [29] and [30] was in fact the case with $n = 1$ (one linear form) or $m = 1$ (simultaneous approximations). Nevertheless, the proofs are valid for the general case. In the case $m = 1$, $n = 1$ Davenport and Schmidt found that the condition $\theta \in \mathfrak{T}_\mu$ with $\mu < 1$ forces θ to be a badly approximable number, that is,

$$\liminf_{q\to\infty} q\|q\theta\| > 0$$

(where q is an integer, of course). In other words, it means that the partial quotients of the continued fraction expansion for θ are bounded. In fact, this result is based on the use of the formula (46).

The proof of Theorem 39 in the case $n=1$ is actually based on the following statement.

Theorem 40 (Schmidt [59]). *Consider a sequence of positive integers N_ν increasing to infinity. For a tuple $\Theta=(\theta^1,\dots,\theta^m)\in\mathbb{R}^m$ consider the lattice*

$$\Lambda(\Theta,N)=\mathscr{A}(\Theta,N)\mathbb{Z}^{m+1},$$

where the matrix $\mathscr{A}(\Theta,N)$ has the form

$$\mathscr{A}(\Theta,N)=\begin{pmatrix} N^{-1} & 0 & 0 & \dots & 0\\ 0 & N^{-1} & 0 & \dots & 0\\ 0 & 0 & N^{-1} & \dots & 0\\ \hdotsfor{5}\\ \theta^1N^m & \theta^2N^m & \theta^3N^m & \dots & N^m\end{pmatrix}.$$

Then for almost all (in the sense of Lebesgue measure) tuples $\Theta\in\mathbb{R}^m$ the sequence of lattices

$$\Lambda(\Theta,N_\nu),\qquad \nu=1,2,3,\dots,$$

is dense in the space of lattices in $\mathbb{R}^{m+1}$ with determinant 1.

The case $m=1$ is connected with lattices of the form

$$\Lambda^*(\Theta,N)=\mathscr{A}^*(\Theta,N)\mathbb{Z}^{n+1},$$

$$\mathscr{A}^*(\Theta,N)=\begin{pmatrix} N^{-1} & 0 & 0 & \dots & 0\\ N^{1/n}\theta_1 & N^{1/n} & 0 & \dots & 0\\ N^{1/n}\theta_2 & 0 & N^{1/n} & \dots & 0\\ \hdotsfor{5}\\ N^{1/n}\theta_n & 0 & 0 & \dots & N^{1/n}\end{pmatrix},\qquad \Theta=\begin{pmatrix}\theta_1\\ \theta_2\\ \vdots\\ \theta_n\end{pmatrix}. \tag{88}$$

In the original paper [25] the result in this case is deduced from the result in the case $n=1$ by means of transference arguments.

In the present paper we do not go into detail about the space of lattices and the convergence in this space. We note only that the classical works on the topic by Mahler [60], [61] are presented in [62], Chap. V. We should also note that a series of results in Diophantine approximation are connected with the consideration of special dynamical systems on the space of lattices. This approach is developed in Kleinbock's papers. In particular, in [57] there is a generalization of the Davenport–Schmidt theorem discussed in this subsection.

7.2. A problem related to successive minima. Consider a lattice $\Lambda\subset\mathbb{R}^d$ and a convex $\mathbf{0}$-symmetric body $\Omega\subset\mathbb{R}^d$. The quantities

$$\mu_l(\Omega,\Lambda)=\inf\{t: t\Omega \text{ contains } l \text{ linearly independent points of } \Lambda\},\qquad 1\leqslant l\leqslant d,$$

are called the *successive minima* of the lattice Λ with respect to the body Ω. The famous second Minkowski convex body theorem (see for example [62], Chap. VIII or [24], Chap. IV) states that

$$\frac{2^d}{d!}\det\Lambda\leqslant\mu_1(\Omega,\Lambda)\cdots\mu_d(\Omega,\Lambda)\cdot\operatorname{meas}\Omega\leqslant 2^d\det\Lambda.$$

In the present subsection we restrict ourselves to the problem of simultaneous approximations: the case $m = 1$. The Minkowski convex body theorem cited at the beginning of the paper can be reformulated as follows in this case. Consider the cube $E = [-1,1]^{n+1} \subset \mathbb{R}^{n+1}$, and for Θ consider the lattice $\Lambda^*(\Theta, N)$ defined in (88). Then for any real $N \geqslant 1$

$$\mu_1\bigl(E, \Lambda^*(\Theta, N)\bigr) \leqslant 1.$$

As pointed out by Schmidt [63], it is easy to see that for any k with $1 \leqslant k \leqslant n$ there exists a sequence of real numbers N_ν tending to infinity such that

$$\mu_k\bigl(E, \Lambda^*(\Theta, N_\nu)\bigr) = \mu_{k+1}\bigl(E, \Lambda^*(\Theta, N_\nu)\bigr).$$

More general results and some applications to problems considered in our paper (in particular, to transference theory) can be found in a recent paper by Schmidt and Summerer [64]. In particular, for $n = 1$ we see from the Minkowski theorem on successive minima that

$$1 \ll \mu_1\bigl(E, \Lambda^*(\Theta, N)\bigr)\, \mu_2\bigl(E, \Lambda^*(\Theta, N)\bigr) \ll 1.$$

Hence, the equality

$$\lim_{N\to+\infty} \mu_1\bigl(E, \Lambda^*(\Theta, N)\bigr) = 0$$

is not possible. But in the case $n > 1$ it can happen that

$$\lim_{N\to+\infty} \mu_{n-1}\bigl(E, \Lambda^*(\Theta, N)\bigr) = 0$$

even for numbers $\theta_1, \dots, \theta_n$ linearly independent together with 1. (One should take Θ from Theorem 4, with a suitable choice of the function ψ.)

In [65] the author proved the following theorem, which answers a question posed by Schmidt in [63].

Theorem 41. *Suppose that $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$. Then there exist real numbers $\Theta = (\theta_1, \dots, \theta_n)$ such that:*

- *the numbers $1, \theta_1, \dots, \theta_n$ are linearly independent over $\mathbb{Z}$;*
- *$\mu_k(E, \Lambda^*(\Theta, N)) \to 0$ as $N \to +\infty$;*
- *$\mu_{k+2}(E, \Lambda^*(\Theta, N)) \to +\infty$ as $N \to +\infty$.*

The proof of the general case is very cumbersome. In the case $k = 1$ it is very simple. In this case the ideas in the proof are close to those in the proof of Theorem 18.

Remark. It is easy to see that the result of Theorem 41 becomes trivial without the assumption of linear independence over $\mathbb{Z}$.

In concluding this subsection we note that an improvement of Theorem 41 was recently announced by Y. Cheung.

8. Transference theorems

Statements that under certain Diophantine conditions on the matrix Θ the transposed matrix ${}^t\Theta$ has some Diophantine properties are known as transference theorems. The simplest transference theorem about singular systems was mentioned in § 6.2 (Theorem 31). Statements about the connection between homogeneous approximations for ${}^t\Theta$ and inhomogeneous approximations for Θ (such as Theorems 27–33) can also be regarded as transference theorems. Many papers have been devoted to transference theorems. Here we give a list of types of transference theorems. This list is by no means complete, but it does give an idea of which mathematicians have been involved with transference theorems and what problems they have considered. A more complete bibliography can be obtained from the papers cited below.

- Classical results by Khintchine on 'ordinary' Diophantine exponents [2], [1]; the general case considered by Dyson [66]; the sharpness of Khintchine's bounds proved by Jarník in [15].
- General constructions due to Mahler [67].
- Transference theorems for singular systems (papers by Jarník [9], Apfelbeck [29], Laurent and Bugeaud [43], [68], [69]; here we also mention a recent paper by Schmidt and Summerer [64]).
- Transference theorems for rational number approximations (Korobov [70], [71]); these results are connected with problems of numerical integration.
- Transference theorems connected with trigonometric sums. Such results can be found in the paper [72] by Gel'fond and the paper [73] by Kashirskii. This approach is based on Siegel's paper [74].
- Transference theorems for products of linear forms (Schmidt and Wang Yuan [75]; see also Kashirskii [73]).
- More precise results involving approximations by rational subspaces (Schmidt [76], Laurent [43], Laurent and Bugeaud [69]).

The classical results are presented in the books by Cassels [23] (Chap. V) and by Schmidt [24] (Chap. IV).

In the present paper we are most interested in transference theorems related to singular systems, that is, transference theorems which deal with the behaviour of quantities of the type

$$\limsup_{t\to+\infty} \varphi(t)\psi_\Theta(t),$$

where $\varphi(t)$ is a certain function and ψ_Θ is Jarník's function (20). Such results include theorems obtained by Jarník and Apfelbeck and recent theorems by Laurent and Bugeaud.

But we start with formulations of classical theorems by Khintchine and Dyson.

First we recall the definition of the exponent $\alpha(\Theta)$ as the supremum of those γ for which

$$\limsup_{t\to+\infty} t^\gamma\psi_\Theta(t) < +\infty,$$

and the definition of the exponent $\beta(\Theta)$ as the supremum of those γ for which

$$\liminf_{t\to+\infty} t^\gamma\psi_\Theta(t) < +\infty$$

(we used these definitions in § 5).

8.1. Theorems of Khintchine and Dyson. A transference theorem for the exponents $\beta(\Theta)$ for a system $\Theta=(\theta^1,\dots,\theta^m)\in\mathbb{R}^m$ and the transposed system ${}^t\Theta$ was proved by Khintchine in his famous paper [1].

Theorem 42. *For a tuple $\Theta=(\theta^1,\dots,\theta^m)\in\mathbb{R}^m$ the following inequalities are valid*:

$$\frac{\beta(\Theta)}{(m-1)\beta(\theta)+m}\leqslant\beta({}^t\Theta)\leqslant\frac{\beta(\theta)-m+1}{m}\,. \tag{89}$$

The result of Dyson [66] is as follows.

Theorem 43. *For any dimensions m and n the following inequality is valid for a matrix Θ*:

$$\beta({}^t\Theta)\geqslant\frac{n\beta(\Theta)+n-1}{(m-1)\beta(\Theta)+m}\,. \tag{90}$$

A simple proof of Dyson's result was given by Khintchine in [8].

Of course, one can deduce Theorem 42 by using Theorem 43 twice (for a row matrix and for a column matrix).

8.2. Results of Jarník and Apfelbeck. Jarník's paper [9] is devoted to transference theorems connecting the cases $m=1$ and $n=1$. We state some general results (Theorems 44, 45; in the original paper [9] they are Satz 7 and Satz 8), then we mention several corollaries. Let us consider a row vector

$$\Theta=(\theta^1,\dots,\theta^m),\qquad m\geqslant 2,$$

and the functions

$$\psi_\Theta(t),\quad \psi_{{}^t\Theta}(t).$$

Theorem 44. *Suppose that $1,\theta^1,\dots,\theta^m$ are linearly independent over $\mathbb{Z}$. Assume that the function $\varphi(t)$ is increasing, the function $t\mapsto t(\varphi(t))^{-1}$ increases to infinity, and*

$$\limsup_{t\to+\infty}\varphi(t)\psi_{{}^t\Theta}(t)<K,$$

where K is a positive constant. Then:

(i) $\limsup_{t\to+\infty}t^{m-1}\varphi(t^m)\psi_\Theta(t)\leqslant m^{2m}K$;

(ii) *if $\varphi(t)\geqslant t^{(m-1)/m}$ for all sufficiently large t and $\rho(t)$ is the inverse function of the function $t\mapsto t(\varphi(t))^{-1}$, then*

$$\limsup_{t\to+\infty}t^{m-2}\rho\biggl(\frac{t}{2K}\biggr)\psi_\Theta(t)\leqslant m^m(K+1).$$

Theorem 45. *Let the numbers $1,\theta^1,\dots,\theta^m$ be linearly independent over $\mathbb{Z}$. Suppose that the function $\varphi(t)$ increases to infinity $+\infty$ as $t\to+\infty$, and let $\rho(t)$ be the function inverse to the function $t\mapsto t\varphi^{(m-1)/m}(t)$. Assume that*

$$\limsup_{t\to+\infty}\varphi(t)\psi_\Theta(t)<K,$$

where K is a positive constant. Then:

(i) $\limsup_{t\to+\infty}\left(\frac{t}{\rho(tK^{(m-1)/m})}\right)^{1/(m-1)}\psi_{{}^t\Theta}(t)\leqslant 3m^2(K+1)$;

(ii) *if $\varphi(t)\geqslant t^{m(2m-3)}$ and for all sufficiently large t the function $\varphi(t)t^{-2m+3}$ is increasing and $\rho_1(t)$ is the function inverse to it, then*

$$\limsup_{t\to+\infty}\left(\frac{t}{\rho_1(tK)}\right)^{1/(m-1)}\psi_{{}^t\Theta}(t)\leqslant 3m^2(K+1).$$

We consider the case $m=2$ separately. In this case

$$\psi_{{}^t\Theta}(t)t^{1/2}\leqslant 1,\qquad \psi_\Theta(t)t^2\leqslant 1$$

by (18). This is why we use the statement (ii) in Theorem 44 and the statement (ii) in Theorem 45.

Corollary 1 ([9], Satz 2). (i) *For the function $\varphi(t)$ suppose that the function $t\mapsto t^{-1}\varphi(t)$ is increasing, $\varphi(t)\geqslant t^2$ for sufficiently large t, and*

$$\limsup_{t\to+\infty}\varphi(t)\psi_\Theta(t)<K.$$

Then for the function $\rho(t)$ inverse to $\varphi(t)$

$$\limsup_{t\to+\infty}\frac{t}{\rho(tK)}\,\psi_{{}^t\Theta}(t)<12(K+1).$$

(ii) *For the function $\varphi(t)$ suppose that the function $t\mapsto t\varphi(t)^{-1}$ is increasing, $\varphi(t)\geqslant t^{1/2}$ for sufficiently large t, and*

$$\limsup_{t\to+\infty}\varphi(t)\psi_{{}^t\Theta}(t)<K.$$

Then for the function $\rho(t)$ inverse to the function $t\mapsto t\varphi(t)^{-1}$

$$\limsup_{t\to+\infty}\rho\left(\frac{t}{2K}\right)\psi_\Theta(t)<4(K+1).$$

The last corollary is much better known as a statement about Diophantine exponents. By the definition of $\alpha(\Theta)$, Corollary 1 leads to the following result.

Corollary 2 ([9], Satz 1). *In the case $m=2$*

$$\alpha(\Theta)=\frac{1}{1-\alpha({}^t\Theta)}\,. \tag{91}$$

Here we should note that in [8] Khintchine gives a short and simple proof of Jarník's equality (91).

We give a corollary of Theorems 44 and 45 relating to the Diophantine exponents $\alpha(\Theta)$ and $\alpha({}^t\Theta)$ in the case of arbitrary m.

Corollary 3 ([9], Satz 3). (i) *The inequalities*

$$\alpha(\Theta) \geqslant (m-1) + m\alpha({}^t\Theta),$$

$$\alpha({}^t\Theta) \geqslant \frac{1}{m-1}\left(1 - \frac{m}{(m-1)\alpha(\Theta)+m}\right) = \frac{\alpha(\Theta)}{(m-1)\alpha(\Theta)+m}$$

hold in any case.

(ii) *If* $\alpha({}^t\Theta) > (m-1)/m$, *then*

$$\alpha(\Theta) \geqslant m-2+\frac{1}{1-\alpha({}^t\Theta)}\,.$$

(ii) *If* $\alpha(\Theta) > m(2m-3)$, *then*

$$\alpha({}^t\Theta) \geqslant \frac{1}{m-1}\left(1-\frac{1}{\alpha(\Theta)-2m+4}\right).$$

Of course, one can get Corollary 2 by putting $m=2$ in Corollary 3.

Apfelbeck [29] generalized Theorems 44 and 45 to the case of arbitrary m and n.

Theorem 46 (Apfelbeck [29]). *Let* Θ *be a non-degenerate matrix, let* K *be a positive number, and let* $\varphi(t)$ *be a function increasing to* $+\infty$ *as* $t\to+\infty$. *Suppose that*

$$\limsup_{t\to+\infty}\varphi(t)\psi_\Theta(t) < K.$$

Then:

(i1) *for* $m=1$

$$\limsup_{t\to+\infty} t^{n-1}\varphi\left(\frac{t^n}{2(n-1)}\right)\psi_{{}^t\Theta}(t) \leqslant 2(n+1)K;$$

(i2) *for* $m>1$

$$\limsup_{t\to+\infty}\left(\frac{t^n}{\rho(K^{(m-1)/(m+n-1)}t)}\right)^{1/(m-1)}\psi_{{}^t\Theta}(t) \leqslant (2(n+m))^{1/(m-1)},$$

where $\rho(t)$ *is the function inverse to the function*

$$t\mapsto \bigl(t^m(\varphi(t))^{m-1}\bigr)^{1/(m+n-1)};$$

(ii) *if for* $m>1$ *the inequality*

$$\varphi(t) \geqslant 2^{m+n-2}Kt^{(2(m+n-2)^2+m-2)/n}$$

holds for sufficiently large t *and the function* $t\mapsto t^{-(2m+n-1)/n}\varphi(t)$ *is increasing, then*

$$\limsup_{t\to+\infty}\left(\frac{t^n}{\rho_1(K^{(m-1)/(m+n-2)}t)}\right)^{1/(m-1)}\psi_\Theta(t) \leqslant 3(m+n),$$

where $\rho_1(t)$ *denotes the function inverse to the function*

$$t\mapsto \bigl(t^{-(m-2)(2m+n-3)/((m-1)n)}\varphi(t)\bigr)^{(m-1)/(m+n-2)}.$$

It is clear that Theorem 46 implies the following corollary concerning the Diophantine exponents $\alpha(\Theta)$ and $\alpha({}^t\Theta)$.

Corollary ([29], Theorem 6). (i) *For any m and n*

$$\alpha({}^t\Theta) \geqslant \frac{n\alpha(\Theta)+n-1}{(m-1)\alpha(\Theta)+m}.$$

(ii) *If $m>1$ and*

$$\alpha(\Theta) > \frac{2(m+n-1)(m+n-3)+m}{n},$$

then

$$\alpha({}^t\Theta) \geqslant \frac{1}{m}\left(n + \frac{n(n\alpha(\Theta)-m)-2n(m+n-3)}{(m-1)(n\alpha(\Theta)-m)+m-(m-2)(m+n-3)}\right).$$

In [29] Apfelbeck proved also that in the case $\alpha(\Theta)=+\infty$ the inequality $\alpha({}^t\Theta) \geqslant n/(m-1)$ (which follows from Theorem 46) is sharp (Theorem 11 in [29], whose proof is based on a construction of singular matrices Θ of a special kind).

8.3. Theorems of Laurent. In [43] Laurent obtained the following result.

Theorem 47. *The following statements are valid for the exponents of two-dimensional Diophantine approximations.*

(i) *For a row vector $\Theta=(\theta^1,\theta^2)\in\mathbb{R}^2$ such that $\dim_{\mathbb{Z}}\Theta=3$ the values*

$$w=\alpha(\Theta), \qquad w^*=\alpha({}^t\Theta), \qquad v=\beta(\Theta), \qquad v^*=\beta({}^t\Theta) \tag{92}$$

(*for the definitions see* § 5.3 *and* § 8.2) *satisfy the relations*

$$2\leqslant w\leqslant +\infty, \qquad w=\frac{1}{1-w^*}, \qquad \frac{v(w-1)}{v+w}\leqslant v^*\leqslant \frac{v-w+1}{w}. \tag{93}$$

(ii) *For any four real numbers (w,w^*,v,v^*) satisfying* (93) *there exists a row vector $\Theta=(\theta^1,\theta^2)\in\mathbb{R}^2$ with $\dim_{\mathbb{Z}}\Theta=3$ such that* (92) *holds.*

We would like to say a few words about Theorem 47. The first relation in (93) is a corollary of the Minkowski Theorem (18). The second is a result of Jarník (Corollary 2 of Theorems 44, 45). The third one was proved by Laurent by means of the best approximations and with the help of Jarník's result (91). The results of Jarník which we formulated as the corollary of Theorem 19 in § 5.1 and the corollary of Theorem 21 in § 5.3 (in the respective cases $m=1$, $n=2$ and $m=2$, $n=1$, of course) can be obtained from the statement (i) of Theorem 47. The statement (ii) of Theorem 47 shows that, in particular, the corollaries of Jarník's theorems in § 5.1 and § 5.3 are sharp (best possible) in the cases $m=1$, $n=2$ and $m=2$, $n=1$. Moreover, it follows from Theorem 47 that the pairs $(\alpha(\Theta),\beta(\Theta))$ and $(\alpha({}^t\Theta),\beta({}^t\Theta))$ can take all admissible (that is satisfying (93)) values.

A generalization of the third inequality in (93) to the case of one linear form in m variables was obtained by Laurent in [68]. We give it here.

Theorem 48. *For a row vector $\Theta = (\theta^1, \dots, \theta^m)$ with $m \geqslant 2$, if $\dim_{\mathbb{Z}} \theta = m+1$, then*

$$\frac{(\alpha(\Theta)-1)\beta(\Theta)}{((m-2)\alpha(\Theta)+1)\beta(\Theta)+(m-1)\alpha(\Theta)} \leqslant \beta({}^t\Theta) \leqslant \frac{(1-\alpha({}^t\Theta))\beta(\Theta)-m+2-\alpha({}^t\Theta)}{m-1}\,. \tag{94}$$

By Jarník's equality (91), in the case $m=2$ the inequalities (94) coincide with the third relation in (93).

From (18) we observe that

$$\alpha(\Theta) \geqslant m, \qquad \alpha({}^t\Theta) \geqslant \frac{1}{m}\,. \tag{95}$$

In the case when both inequalities in (95) turn into equalities we see that Laurent's inequalities (94) coincide with Khintchine's inequalities (89). But if the inequalities (95) are strict, then the inequalities (94) are stronger than the inequalities (89). Thus, Theorem 48 asserts that Theorem 42 can be improved if the matrix Θ is singular. The author does not know any similar improvement of Dyson's Theorem 43 connected with singular matrices Θ. It is clear that such a result must hold. Certain improvements of Laurent's results are due to Laurent and Bugeaud [69].

9. Hausdorff dimension of sets of singular systems

Not very much is known about the exact values and bounds for the Hausdorff dimension of sets of singular matrices. We gather some such results below. Several problems concerning the determination of Hausdorff dimensions were posed by Laurent and Bugeaud in [77]. To the author's knowledge, most papers deal with the case $n=1$, and only estimates of the Hausdorff dimension are obtained. An exception is a paper by Cheung in which the following result is obtained.

Theorem 49 (Cheung [78]). *The Hausdorff dimension of the set of singular systems (in the sense of Khintchine's definition for $n=2$, $m=1$) of the form $\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$ is equal to* $4/3$.

From the transference principle it follows that the result of Theorem 49 remains true for singular systems (θ^1, θ^2) with $n=1$ and $m=2$.

Below we shall consider sets $E_m(\alpha)$ consisting of singular vectors $\Theta = (\theta^1, \dots, \theta^m)$ (that is, the case of arbitrary $m \geqslant 2$ and $n=1$) and defined as follows:

$$E_m(\alpha) = \Bigl\{\Theta \in \mathbb{R}^m : \lim_{t\to+\infty} t^\alpha \psi_\Theta(t) = 0\Bigr\}.$$

We now give the sharpest upper and lower bounds for Hausdorff dimension known to the author.

The following result is due to Baker [79]. It improves earlier results of Baker [80], Yavid [81], and Rynne [82] (Yavid was the first to construct a counterexample to Baker's conjecture in [80] stating that the Hausdorff dimension of the set of singular linear forms should be very small, but Yavid's result was *quantitatively* improved by Rynne).

Theorem 50 (Baker [79]). *Suppose that $m \geqslant 3$ and $\alpha > m$. Then the Hausdorff dimension of the set $E_m(\alpha)$ has the lower bound*

$$\mathrm{HD}\, E_m(\alpha) \geqslant m - 2 + \frac{m}{\alpha}\,.$$

An upper bound was obtained by Rynne [83], who improved Baker's result in [80].

Theorem 51 (Rynne [83]). *Suppose that $m \geqslant 3$ and $\alpha > m$. Then the Hausdorff dimension of the set $E_m(\alpha)$ has the upper bound*

$$\mathrm{HD}\, E_m(\alpha) \leqslant m - 2 + \frac{2m+2}{\alpha+1}\,.$$

In the case $m = 2$, both of the best known bounds (upper bound in [79], lower bound follows from [80]) were due to Baker for some time.

Theorem 52 (Baker [80], [79]). *The Hausdorff dimension of the set $E_2(\alpha)$ has the bounds*

$$\frac{2}{\alpha} \leqslant \mathrm{HD}\, E_2(\alpha) \leqslant \frac{6}{\alpha+1}\,.$$

The lower bound in this theorem seems to be the best one so far. Using Jarník's equality (91), the inequality (58) in the corollary of Theorem 19 (also due to Jarník), and a particular case ($n = 2$, $m = 1$) of a result of Dodson in [84] (Theorem 53 below), Bugeaud and Laurent deduce in [77] the inequality

$$\mathrm{HD}\, E_2(\alpha) \leqslant \frac{3\alpha}{\alpha^2 - \alpha + 1}\,,$$

which is stronger than the upper estimate in Theorem 52. More precisely, we should say that the following inequality was proved in [77]:

$$\mathrm{HD}\{\Theta \in \mathbb{R}^2,\ \alpha(\Theta) \geqslant \alpha\} \leqslant \frac{3\alpha}{\alpha^2 - \alpha + 1}\,.$$

We now give the general result in [84] which we used above. Recall that the exponent $\beta(\Theta)$ was defined as the supremum of those γ, for which

$$\liminf_{t \to +\infty} t^{\gamma} \psi_{\Theta}(t) < +\infty.$$

Theorem 53. *For any $\tau > m/n$ the Hausdorff dimension of the set*

$$\mathfrak{W}_{m,n}(\tau) = \{\Theta : \beta(\Theta) \geqslant \tau\}$$

is equal to

$$\mathrm{HD}\, \mathfrak{W}_{m,n}(\tau) = (m-1)n + \frac{m+n}{\tau+1}\,.$$

We give two simple corollaries of the above results. Theorems 50 and 51 imply the following result about the Hausdorff dimension of the set

$$E_m(\infty) = \bigcap_{\alpha > m} E_m(\alpha).$$

Corollary 1. *The Hausdorff dimension of the set $E_m(\infty)$ is*

$$\operatorname{HD} E_m(\infty) = m - 2.$$

We note also that Theorem 50 implies the following statement.

Corollary 2. *The Hausdorff dimension of the set of singular systems (in the sense of Khintchine's definition) in the case $n = 1$ and for arbitrary $m \geqslant 2$ is bounded from below by $m - 1$.*

In concluding this section we cite the books [85] by Bernik and Mel'nichuk and [86] by Bernik and Dodson dealing with Diophantine approximation and Hausdorff dimension.

10. Approximations with non-negative integers

10.1. Two-dimensional approximations. Put $\tau = (1+\sqrt{5}\,)/2$. In [87] Schmidt obtained the following result.

Theorem 54. *Let the numbers θ^1 and θ^2 be linearly independent together with 1 over $\mathbb{Z}$. Then there is a sequence of two-dimensional integer vectors $(x_1(i), x_2(i))$ such that:*

1) $x_1(i), x_2(i) > 0$;
2) $\|\theta^1 x_1(i) + \theta^2 x_2(i)\|(\max\{x_1(i), x_2(i)\})^\tau \to 0$ *as* $i \to +\infty$.

A well-known conjecture (see [87], [63]) supposes that the exponent τ in Theorem 54 can be replaced by $2 - \varepsilon$ for arbitrarily small ε. This conjecture has not yet been proved, but there are several results by several mathematicians which improve and generalize Schmidt's result (for example, see [88]–[90]). Schmidt's proof uses the assertion of linear independence of the best approximations of a linear form (Theorem 7, $m = 2$, $n = 1$), and actually involves the consideration of two cases depending on the type of singularity of the system θ^1, θ^2 $(n = 1,\ m = 2)$. These two cases are as follows. Considering the set of indices ν for which the vectors $\mathbf{z}_{\nu-1}$, $\mathbf{z}_\nu$, $\mathbf{z}_{\nu+1}$ are linearly independent, we suppose in the first case that there are infinitely many ν such that $\zeta_\nu \leqslant M_\nu^{1/\tau} M_{\nu+1}^{-\tau/(\tau-1)}$. In the second case we suppose that there exist infinitely many ν such that the opposite inequality holds.

In [91] the author considers the case of a badly approximable system θ^1, θ^2. We state the result in [91]. For a real number $\gamma \geqslant 2$ let

$$g(\gamma) = \tau + \frac{2\tau - 2}{\tau^2\gamma - 2}\,.$$

Obviously $g(\gamma)$ is strictly decreasing, $g(2) = 2$, and $\lim_{\gamma\to+\infty} g(\gamma) = \tau$. For positive Γ let

$$C(\Gamma) = 2^{18}\Gamma^{(\tau-\tau^2)/(\tau^2\gamma-2)}.$$

Theorem 55. *Suppose that the real numbers θ^1 and θ^2 are badly approximable in the sense that for some $\Gamma \in (0,1)$ and some $\gamma \geqslant 2$ the inequality*

$$\|\theta^1 m_1 + \theta^2 m_2\| \geqslant \frac{\Gamma}{(\max\{|m_1|, |m_2|\})^\gamma} \tag{96}$$

holds for all integer vectors $(m_1, m_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Then there exists an infinite sequence of two dimensional integer vectors $(x_1(i), x_2(i))$ such that:
1) $x_1(i), x_2(i) > 0$;
2) $\|\theta^1 x_1(i) + \theta^2 x_2(i)\|(\max\{x_1(i), x_2(i)\})^{g(\gamma)} \leqslant C(\Gamma)$ *for all i.*

We do not dwell on the proof of this theorem. It follows original ideas of Schmidt and is presented in sufficient detail in [91].

10.2. Linear forms in $k > 2$ variables. In the case $k \geqslant 3$ a linear form in k variables may not take small values for positive x_j in general. Here we give a result due to Schmidt in [87].

Theorem 56. *There exists a vector $\Theta = (\theta_1, \dots, \theta_k)$, $k \geqslant 3$, such that:*
- $\dim_{\mathbb{Z}} \Theta = k+1$;
- *for any $\varepsilon > 0$ there exists a $c(\varepsilon) > 0$ such that for all positive integers $x_1, \dots, x_k$*

$$\|\theta_1 x_1 + \dots + \theta_k x_k\| > c(\varepsilon)\Bigl(\max_{1 \leqslant i \leqslant k} |x_i|\Bigr)^{-2-\varepsilon}.$$

The proof of Theorem 56 uses a result of Davenport and Schmidt in [92] which ensures the existence of real numbers with anomaly simultaneous approximations.

Theorem 57. *Suppose that $n \geqslant 2$ and let $\psi(t)$ be a positive function of the real variable t. Then there exists a vector $\Theta = (\theta_1, \dots, \theta_n)$ such that:*
- $\dim_{\mathbb{Z}} \Theta = n+1$;
- *for sufficiently large t there is a positive integer $q \leqslant t$ such that*

$$\|q\theta_j\| \leqslant \psi(t) \tag{97}$$

for all $1 \leqslant j \leqslant n$ with the possible exception of one index $j_0 = j_0(t)$.

We note that in the case $n \geqslant 2$ and $\psi(t) = O(t^{-1})$ (as $t \to +\infty$) one cannot ensure the inequalities (97) for all $1 \leqslant j \leqslant n$. This would contradict Theorem 17.

Following Schmidt [87], we show that Theorem 57 implies Theorem 56 by means of simple transference arguments. We take $n = k-1$ and $\psi(t) = e^{-t}$, and for the numbers $\theta_1, \dots, \theta_{k-1}$ in Theorem 57 we find a number θ_k such that with some positive constant $c_1(\varepsilon)$ the inequality

$$\|\theta_i u + \theta_k v\| > c_1(\varepsilon)(|u| + |v|)^{-2-\varepsilon}, \qquad 1 \leqslant i \leqslant k-1,$$

is valid for all integers u and v with $v \neq 0$. (Standard arguments involving the Borel–Cantelli lemma show that it is valid for *almost all* numbers θ_k. These arguments are close to those used in §§ 2.2, 2.3 above.)

For an integer point $(x_1, \dots, x_k)$ with sufficiently large $M = \max_{1 \leqslant i \leqslant k} |x_i|$ we now put $t = (\log M)^2$ and choose q with $1 \leqslant q \leqslant t$ to satisfy the conclusion of

Theorem 57. Without loss of generality assume that $j_0(t) = 1$. Hence if $x_k \neq 0$, then

$$\begin{aligned}\|\theta_1 x_1 + \cdots + \theta_k x_k\| &\geqslant q^{-1}\|\theta_1 q x_1 + \cdots + \theta_k q x_k\| \\ &\geqslant q^{-1}\bigl(\|\theta_1 q x_1 + \theta_k q x_k\| - M(\|\theta_2 q\| + \cdots + \|\theta_{k-1} q\|)\bigr) \\ &> q^{-1}\biggl(c_1\biggl(\frac{\varepsilon}{3}\biggr)(qM)^{-2-\varepsilon/3} - kMe^{-t}\biggr) > M^{-2-\varepsilon}.\end{aligned}$$

Theorem 56 is proved.

11. Kozlov's problem

Consider real numbers $\Theta = (\theta^1, \dots, \theta^m)$ linearly independent over $\mathbb{Z}$ and a real-valued function $f(x_1, \dots, x_m)$ that is periodic in each variable x_i with period 1 and is sufficiently smooth (for example, continuous). Everywhere in this section we assume the zero mean value condition

$$\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_m)\, dx_1 \cdots dx_m = 0. \tag{98}$$

According to a famous theorem of Weyl [93], for a continuous function f and for any initial phase $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$ one has

$$I(T, \mathbf{y}) = I^{[f,\Theta]}(T, \mathbf{y}) = \int_0^T f(\theta^1 t + y_1, \dots, \theta^m t + y_m)\, dt = o(T), \qquad T \to +\infty.$$

An interesting problem about recurrence and oscillation of the integral $I(T, \mathbf{y})$ was posed by Kozlov.

We say that the integral $I(T, \mathbf{y})$ *oscillates* (for a given value of the initial phase $\mathbf{y}$) if both of the sets

$$\mathfrak{T}_+ = \{T \in \mathbb{R}_+ \mid I(T, \mathbf{y}) > 0\}, \qquad \mathfrak{T}_- = \{T \in \mathbb{R}_+ \mid I(T, \mathbf{y}) < 0\}$$

are unbounded.

We say that the integral $I(T, \mathbf{y})$ *has the recurrence property* (for a given initial phase $\mathbf{y}$) if

$$\liminf_{T \to +\infty} |I(T, \mathbf{y})| = 0.$$

The continuous setting described above has a discrete analogue.

Consider a collection of real numbers $\Theta = (\theta^1, \dots, \theta^m)$ linearly independent together with 1 over $\mathbb{Z}$. Suppose that the real function $F(x_1, \dots, x_m)$ is sufficiently smooth and 1-periodic in each variable x_i. Assume the zero mean value condition

$$\int_0^1 \cdots \int_0^1 F(x_1, \dots, x_m)\, dx_1 \cdots dx_m = 0. \tag{99}$$

Weyl's theorem on equality of the space mean value and time mean value cited above asserts in the discrete situation that for any initial phase $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$

$$S(Q, \mathbf{y}) = S^{[F,\Theta]}(Q, \mathbf{y}) = \sum_{s=1}^{Q} F(\theta^1 s + y_1, \dots, \theta^m s + y_m)\, dt = o(Q), \qquad Q \to +\infty.$$

The definitions of the oscillation property and the recurrence property for the sum $S(Q,\mathbf{y})$ as $Q \to +\infty$ are analogous to those for the integral $I(T,\mathbf{y})$.

For a function $f(x_1,\dots,x_m)$ of m variables one can consider the function $F_z(y_1,\dots,y_{m-1})$ of $m-1$ variables $y_1,\dots,y_{m-1}$ (here z is treated as a parameter) defined as

$$F_z(y_1,\dots,y_{m-1}) = \int_0^{1/\theta^m} f(\theta^1 t + y_1,\dots,\theta^{m-1}k + y_{m-1}, \theta^m t + z)\,dt,$$

and the new set of frequencies $\Theta^* = (\theta^1/\theta^m,\dots,\theta^{m-1}/\theta^m)$ (under the specified conditions these numbers are linearly independent *together with* 1 over $\mathbb{Z}$). Then the recurrence or oscillation property of the sum $S^{[F_z,\Theta]}(Q,\mathbf{y})$ with some value of z and with some initial phase $\mathbf{y} = (y_1,\dots,y_{m-1})$ implies the recurrence or oscillation property of the integral $I(T,\mathbf{x})$ for the corresponding value of the initial phase $\mathbf{x}$. In general the converse statement may not be true.

11.1. Peres' lemma and Halász' theorem. In this subsection we state two general results from ergodic theory and discuss their applications to the problem under consideration.

By a *dynamical system* here we mean a probability space $(\Omega,\mathscr{A},\mu)$ with some μ-preserving ergodic transformation T of the set Ω into itself (the basic notions of ergodic theory can be found in [94]). The Birkhoff ergodic theorem says that for an integrable function g from Ω to $\mathbb{R}$

$$\frac{1}{Q}\sum_{s=1}^{Q} g(T^s x) \to \int_\Omega g\,d\mu, \qquad Q \to +\infty,$$

for almost all $x \in \Omega$.

The following statement is known as Peres' Lemma.

Lemma 6 (Peres [95]). *Suppose that Ω is compact and T is continuous. Then for any continuous function g from Ω to $\mathbb{R}$ there exists an $x \in \Omega$ such that*

$$\frac{1}{Q}\sum_{s=1}^{Q} g(T^s x) \geqslant \int_\Omega g\,d\mu.$$

In our problem Lemma 6 has the following consequence.

Corollary. *Suppose that Θ consists of frequencies linearly independent together with 1 over $\mathbb{Z}$. Assume that $F(x_1,\dots,x_m)$ is a continuous function with period 1 in each variable and zero mean value. Then there exists a $\mathbf{y} \in [0,1)^m$ such that*

$$S^{[F,\Theta]}(Q,\mathbf{y}) \geqslant 0 \quad \forall\, Q \in \mathbb{N}.$$

An analogous statement was obtained for the integral $I^{[f,\Theta]}(T,\mathbf{y})$ by Bohl [96]. A stronger result is due to Kozlov [97], [98].

Theorem 58 (Kozlov [97], [98]). (i) *Let the frequencies $\theta^1,\dots,\theta^m$ be linearly independent over $\mathbb{Z}$, and let the periodic continuous function f have zero mean value. Then there exists a $\mathbf{y} \in [0,1)^m$ such that*

$$I^{[f,\Theta]}(T,\mathbf{y}) \geqslant 0 \quad \forall\, T \in \mathbb{R}$$

and moreover, $f(\mathbf{y}) = 0$.

(ii) *In the case when the frequencies* $\theta^1,\dots,\theta^m$ *are linearly dependent over* $\mathbb{Z}$ *there exist at least two different points* $\mathbf{y}_1,\mathbf{y}_2\in[0,1)^m$ *satisfying the conclusion of the statement* (i).

We now formulate one of the results in the paper [99] of Halász, relating to the behavior of Birkhoff sums for a function g integrable over Ω.

Theorem 59 (Halász [99]). *For any integrable* g *the difference*

$$\sum_{s=1}^{Q} g(T^s x) - Q\int_\Omega g\,d\mu \tag{100}$$

changes sign in a weak sense infinitely many times for almost all $x\in\Omega$.

A quantity is said to change sign *in a weak sense* infinitely many times if it cannot eventually be strictly positive or strictly negative.

Corollary. *For any tuple* Θ *of frequencies linearly independent together with* 1 *over* $\mathbb{Z}$ *and for any integrable function* $F(x_1,\dots,x_m)$ 1*-periodic in each variable and with zero mean value, the sum* $S^{[F,\Theta]}(Q,\mathbf{y})$ *changes sign in a weak sense infinitely many times for almost all values of the initial phase* $\mathbf{y}$.

11.2. Individual recurrence. In the case $m=2$ the recurrence property of the integral $I^{[f,\Theta]}(T,\mathbf{y})$ for a periodic function $f(x_1,x_2)\in C^2([0,1]^2)$ with zero mean value was proved by Kozlov himself for an arbitrary pair $\Theta=(\theta^1,\theta^2)$ of frequencies with $\theta^1/\theta^2\notin\mathbb{Q}$ and for an arbitrary initial phase (y_1,y_2) in [97], [98]. In these papers he noted that in the case $m=2$ a more precise statement about the recurrence property of the integral $I(T,\mathbf{y})$ is true: namely, there exists a sequence of positive *integers* T_ν such that simultaneously

$$\max_{1\leqslant i\leqslant 2}\|\theta^i T_\nu\|\to 0,\quad I(T_\nu,\mathbf{y})\to 0,\qquad \nu\to+\infty. \tag{101}$$

He also observed that if $f(y_1,y_2)\neq 0$, then the 'strong recurrence property' (101) implies that the integral $I(T,\mathbf{y})$ oscillates for this value $\mathbf{y}=(y_1,y_2)$ of the initial phase.

These results of his were improved by Sidorov [100] (detailed comments will be given in the next subsection). In the case of an arbitrary dimension m, results on oscillation and recurrence were obtained by Konyagin for an *odd* function f (see [101]) and by the author in the general case (see [102], [103]). Here we give these results and make some comments.

First of all we state two results in [102] and [103].

Theorem 60. *Let* $m\geqslant 2$, *and let the function* f *belong to the class* $C^w([0,1]^m)$ *with*

$$w=w(m)=[\exp(20m\log m)]. \tag{102}$$

Assume the zero mean value condition (98). *Then for any tuple* $\Theta=(\theta^1,\dots,\theta^m)$ *of frequencies linearly independent over* $\mathbb{Z}$ *and for any initial phase* $\mathbf{y}$ *the integral* $I(T,\mathbf{y})$ *has the recurrence property.*

Theorem 61. *Under the conditions of Theorem 60 suppose that for the initial phase* $\mathbf{y} = (y_1, \ldots, y_m)$

$$f(y_1, \ldots, y_m) \neq 0.$$

Then the integral $I(T, \mathbf{y})$ *oscillates.*

Theorem 60 is proved in the papers [102] and [103]. However, in these papers the theorem is formulated with stronger conditions on the smoothness exponent $w(m)$. Nevertheless, the proof in [102] and [103] works for the exponent (102) in the theorem as stated above. Unfortunately, an explicit calculation is found only in the author's D.Sc. thesis [104], but it can easily be reconstructed. Though Theorem 61 is not a direct corollary of Theorem 60, its proof is almost identical to that of Theorem 60. A precise argument is also given in [104].

From Theorem 60 we deduce the following statement as an obvious corollary.

Corollary. *Suppose that a function* $F(\mathbf{x})$ *with zero mean value* (99) *belongs to the class* $C^{w(m)}([0,1]^m)$. *Then for any frequencies* $\Theta = (\theta^1, \ldots, \theta^m)$ *that are linearly independent together with* 1 *over* $\mathbb{Z}$ *and for any initial phase* $\mathbf{x}$,

$$\liminf_{Q \to +\infty} |S(Q, \mathbf{x})| < +\infty.$$

It is clear that the corollary does not assert the recurrence property for the sum $S(Q, \mathbf{y})$. This question seems to be open for $S(Q, \mathbf{y})$.

Here we should note that the proofs of Theorems 60 and 61 are connected with the investigation of the distribution of the best Diophantine approximations of a linear form (the case $n = 1$) and with a detailed analysis of the oscillation of harmonics in the Fourier series expansion for the function f.

We now state Konyagin's result in [101].

Theorem 62. *Let* $m \geqslant 3$. *Suppose that a periodic function* f *belongs to the class* $C^{w_1}([0,1]^m)$, *where*

$$w_1 = w_1(m) = 3m \cdot 2^{m-1}. \tag{103}$$

Assume the zero mean value condition (98), *and in addition suppose that for all* $\mathbf{x}$

$$f(-x_1, \ldots, -x_m) = -f(x_1, \ldots, x_m). \tag{104}$$

Then the integral $I(T, \mathbf{0})$ *has the recurrence property.*

Note that $w_1(m) < w(m)$, and hence in Konyagin's theorem dealing with an odd function the smoothness condition on f is weaker than in Theorem 60.

We remark that Theorem 62 can be generalized in the following way. Instead of one function f satisfying the conditions of Theorem 62 we can consider a *finite collection* of functions $f_1, \ldots, f_r$ such that each satisfies the conditions of Theorem 62. Then

$$\liminf_{T \to +\infty} \sum_{1 \leqslant j \leqslant r} |I^{[f_j, \Theta]}(T, \mathbf{0})| = 0.$$

The proof of this assertion is a direct generalization of the proof in [101].

We now make a few remarks about the smoothness of f sufficient for the recurrence property.

Using a development of Treshchev's ideas and a generalization of an example of Poincaré in [106], [107], the author (see [105]) proved the following statement.

Theorem 63. *Suppose that the tuple* $\Theta = (\theta^1, \dots, \theta^m)$ *of frequencies is badly approximable in the sense that for some positive number* $\Gamma = \Gamma(\Theta)$ *and all non-zero integer vectors* $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$

$$|k_1\theta^1 + \dots + k_m\theta^m| > \Gamma \cdot \Bigl(\max_{1 \leqslant i \leqslant m} |k_i|\Bigr)^{1-m}.$$

Then there exists a function $f^{[\Theta]}(\mathbf{x})$ *in* $C^{m-2}([0,1]^m) \setminus C^{m-1}([0,1]^m)$ *that is* 1*-periodic in each variable, has zero mean value, and is such that*

$$\lim_{T \to +\infty} I^{[f^{[\Theta]}, \Theta]}(T, \mathbf{0}) = +\infty.$$

We note that every badly approximable tuple consists of numbers linearly independent over $\mathbb{Z}$.

On the other hand, Konyagin constructed a certain singular system Θ of frequencies (see [101]) to get the following result.

Theorem 64. *Let* $m \geqslant 4$, *and let*

$$w_3 = w_3(m) = \left[\frac{2^{m-1}(m-2)^{m-2}}{(m-1)^{m-1}}\right] - 1.$$

Then there exist a periodic function $f \in C^{w_3}([0,1]^m)$ *and a tuple of frequencies* $\Theta = (\theta^1, \dots, \theta^m)$ *consisting of numbers linearly independent over* $\mathbb{Z}$ *such that* (104) *holds and*

$$\lim_{T \to +\infty} I^{[f^{[\Theta]}, \Theta]}(T, \mathbf{0}) = +\infty.$$

We note that for $m \geqslant 9$ Theorem 64 gives an example of a function f without the recurrence property of the integral $I^{[f^{[\Theta]}, \Theta]}(T, \mathbf{0})$ and yet more smooth than the function in Theorem 63.

In [102] there is a negative result on the existence of simultaneous recurrence for the integrals in the general case. It is also related to singular systems Θ. We give it here.

Theorem 65. *Suppose that the function* $\Phi(t)$ *decreases to zero* (*arbitrarily fast*). *Then there exist two real-valued functions*

$$f_j(x_1, x_2, x_3) = \sum_{(k_1,k_2,k_3) \in \mathbb{Z}^3 \setminus \{\mathbf{0}\}} f_{j;k_1,k_2,k_3} \exp\bigl(2\pi i(k_1x_1 + k_2x_2 + k_3x_3)\bigr)$$

$(j = 1, 2)$ *such that their Fourier coefficients have the estimates*

$$|f_{j;k_1,k_2,k_3}| \leqslant \Phi\Bigl(\max_{1 \leqslant j \leqslant 3} |k_j|\Bigr) \quad \forall\, k \in \mathbb{Z}^3, \quad j = 1, 2,$$

but at the same time for some triple $\Theta = (\theta^1, \theta^2, \theta^3)$ *of real numbers linearly independent over* $\mathbb{Z}$

$$\liminf_{T \to +\infty} \sum_{1 \leqslant j \leqslant 2} |I^{[f_j, \Theta]}(T, \mathbf{0})| = +\infty.$$

We refer here to the author's paper [108], where a partial solution of the problem was given in the case $m = 3$.

11.3. Uniform recurrence. By *uniform recurrence* we mean recurrence of the quantities

$$J(T) = J^{[f,\Theta]}(T) = \sup_{\mathbf{y}\in\mathbb{R}^m} |I^{[f,\Theta]}(T,\mathbf{y})|, \quad R(Q) = R^{[f,\Theta]}(Q) = \sup_{\mathbf{y}\in\mathbb{R}^m} |S^{[F,\Theta]}(Q,\mathbf{y})|.$$

Sidorov's result cited above deals with just this kind of recurrence. Here we give the exact formulation.

Theorem 66 (Sidorov [100]). *Let $m = 1$. Let $F(x)$ be a 1-periodic absolutely continuous function of the real argument x, and let $\theta \notin \mathbb{Q}$. Then the sum $S^{[F,\Theta]}(Q, y)$ has the uniform recurrence property, that is,*

$$\liminf_{Q\to+\infty} R(Q) = 0.$$

Sidorov's proof is based on continued fractions arguments.

As was observed in [109], the case $m > 1$ differs radically from the case $m = 1$. In [109] the author proved that uniform recurrence can be absent even in the case $m = 2$ and even for smooth $F(x_1, x_2)$. For this purpose he used a singular system (θ_1, θ_2) and re-proved Khintchine's theorem (Theorem 1, 1926), which he did not know about at the time. Here we give a more general result of Kolomeikina and the author in [110], along with the main auxiliary result there on the existence of singular systems Θ of a special kind with $n = 1$.

Consider a Fourier series expansion

$$F(x_1, \dots, x_m) = \sum_{(k_1,\dots,k_m)\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} F_{k_1,\dots,k_m} \exp\bigl(2\pi i(k_1x_1 + \dots + k_mx_m)\bigr).$$

By the *spectrum of the function F* we mean the set

$$\operatorname{spec} F = \{(k_1, \dots, k_m) \in \mathbb{Z}^m : \ F_{k_1,\dots,k_m} \neq 0\}.$$

The following result shows that there may be no uniform recurrence property even under a very strong smoothness condition on the function F.

Theorem 67. *Suppose that a function F with period 1 in each variable belongs to the class $C^1([0, 1]^m)$ and satisfies (99).*

Consider the following two conditions.

(A) *For any positive-valued function $\lambda(t) = o(t)$, $t \to +\infty$, there exists a tuple $\Theta = (\theta^1, \dots, \theta^m)$ of real numbers linearly independent together with 1 over $\mathbb{Z}$ such that*

$$R^{[F,\Theta]}(Q) > \lambda(Q)$$

for sufficiently large Q.

(B) *There exist an $R > 0$ and a non-zero integer point $p = (p_1, \dots, p_m) \in \mathbb{Z}^m$ such that $\operatorname{spec} F \subset \mathscr{B}(R) \cup \mathscr{L}(p)$, where $\mathscr{B}(R)$ is the ball in $\mathbb{R}^m$ with zero centre and radius R, and $\mathscr{L}(p)$ is the line in $\mathbb{R}^m$ passing through zero and p.*

Then the condition (A) is equivalent to the negation of the condition (B).

The proof of Theorem 67 is based on the following lemma.

Lemma 7. *Assume the negation of the condition (B) in Theorem 67. Then for any function $\psi(t)$ decreasing to zero as $t \to +\infty$ there exist a vector $\Theta = (\theta^1, \dots, \theta^m)$ of numbers linearly independent together with 1 over $\mathbb{Z}$ and an infinite sequence of integer vectors*

$$(k_{1,\nu}, \dots, k_{m,\nu}) \in \operatorname{spec} F, \qquad \nu = 1, 2, 3, \dots,$$

such that for all ν

$$\|\theta^1 k_{1,\nu} + \dots + \theta^m k_{m,\nu}\| \leqslant \psi\Bigl(\max_{1 \leqslant i \leqslant m} |k_{i,\nu+1}|\Bigr). \tag{105}$$

It is clear that the condition (105) looks like the condition (19) in the equivalent definition of ψ-singularity. In some sense Lemma 7 deals with the 'best approximations in the set $\operatorname{spec} F$'.

We note that a result on uniform recurrence can easily be obtained if the vector Θ is not ψ-singular (for a certain choice of ψ). The following statement holds.

Proposition 4. *Suppose that the function $\Phi(t)$ decreases to zero as $t \to +\infty$ and that the series*

$$\sum_{k_1, \dots, k_m = -\infty}^{+\infty} \Phi\Bigl(\max_{1 \leqslant i \leqslant m} |k_i|\Bigr) \max_{1 \leqslant i \leqslant m} |k_i|$$

converges. Assume that the Fourier coefficients of the function $F(x_1, \dots, x_m)$ have the upper estimate

$$|F_{k_1, \dots, k_m}| \leqslant \Phi\Bigl(\max_{1 \leqslant i \leqslant m} |k_i|\Bigr)$$

and that the zero mean condition (99) holds. Consider the function

$$\Phi_1(t) = \sum_{(k_1, \dots, k_m) \colon \max_{1 \leqslant i \leqslant m} |k_i| \geqslant t} \Phi\Bigl(\max_{1 \leqslant i \leqslant m} |k_i|\Bigr),$$

which tends to zero.

Let $n = 1$ and suppose that the vector $\Theta = (\theta^1, \dots, \theta^m)$ of real numbers linearly independent together with 1 over $\mathbb{Z}$ does not form a ψ-singular system for any function $\psi(t)$ such that

$$\psi(t)\bigl(\Phi_1(t)\bigr)^{-1/m} \to +\infty, \qquad t \to +\infty.$$

Then there exists a sequence of integers q_μ ($\mu = 1, 2, 3, \dots$) such that

$$R^{[F,\Theta]}(q_\mu) \to 0, \qquad \max_{1 \leqslant i \leqslant m} \|\theta^i q_\mu\| \to 0, \qquad \mu \to +\infty.$$

Here we make a comment. For example, the conditions of Proposition 4 will be satisfied under the following assumptions: the Fourier coefficients of F satisfy the condition

$$|F_{k_1, \dots, k_m}| \leqslant \Gamma\Bigl(\max_{1 \leqslant i \leqslant m} |k_i|\Bigr)^{-\gamma} \quad \forall\, \mathbf{k} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$$

for some $\Gamma > 0$ and $\gamma > (m+1)m$, and the vector Θ satisfies

$$\limsup_{t\to+\infty} \psi_\Theta(t)t^{\gamma/m-1} = +\infty.$$

It is clear that Proposition 4 ensures the 'strong recurrence property' (that is, both assertions in (101) hold simultaneously). Moreover, it ensures the uniform recurrence property. It is obvious from the proof that the analogue of Proposition 4 is valid for the simultaneous recurrence property of integrals for any finite collection of functions. From Proposition 4 it becomes clear that in the proofs of Theorems 60 and 61 the main difficulty is related to the consideration of the case when Θ is a ψ-singular system with a function ψ rapidly decreasing to zero.

Proof of Proposition 4. If ψ-singularity fails, then by Proposition 1 there exists an infinite subsequence ν_μ such that for the best approximations we have

$$\zeta_{\nu_\mu} \cdot \bigl(\Phi_1(M_{\nu_\mu+1})\bigr)^{-1/m} \to +\infty, \qquad \mu \to +\infty. \tag{106}$$

It is clear that

$$\lambda_{\nu_\mu} = \zeta_{\nu_\mu} \cdot \Phi_1(M_{\nu_\mu+1}) \to 0, \qquad \mu \to +\infty.$$

Put

$$Q_\mu = \bigl(\zeta_{\nu_\mu} \cdot \Phi_1(M_{\nu_\mu+1})\bigr)^{-m/(m+1)}.$$

By Dirichlet's theorem let q_μ be a positive integer such that

$$1 \leqslant q_\mu \leqslant Q_\mu, \qquad \max_{1\leqslant i\leqslant m} \|\theta_i q_\mu\| \leqslant Q_\mu^{-1/m}.$$

Then

$$q_\mu \to +\infty, \quad \max_{1\leqslant i\leqslant m} \|\theta_i q_\mu\| \to 0, \qquad \mu \to +\infty.$$

On the other hand,

$$\begin{aligned}
R^{[F,\Theta]}(q_\mu) &\leqslant \sum_{\mathbf{k}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}} |F_{k_1,\dots,k_m}| \left|\sum_{s=1}^{q_\mu} \exp\bigl(2\pi i(k_1\theta^1 + \dots + k_m\theta^m)s\bigr)\right| \\
&\ll \sum_{\mathbf{k}\colon \max_i |k_i| < M_{\nu_\mu+1}} \Phi\Bigl(\max_i |k_i|\Bigr) \frac{\max_i |k_i| \cdot \max_i \|\theta_i q_\mu\|}{\|k_1\theta^1 + \dots + k_m\theta^m\|} + q_\mu \Phi_1(M_{\nu_\mu+1}) \\
&\ll \frac{1}{\zeta_{\nu_\mu} Q_\mu^{1/m}} + Q_\mu \Phi_1(M_{\nu_\mu+1}) = 2\Bigl(\zeta_{\nu_\mu}^m \bigl(\Phi_1(M_{\nu_\mu+1})\bigr)^{-1}\Bigr)^{-1/(m+1)} \to 0, \\
&\qquad \mu \to +\infty.
\end{aligned}$$

In the last inequality we used (106). The proposition is proved.

12. Singular systems of a special kind

Among the results discussed in this paper there are existence theorems for singular systems of a special kind. As examples we mention Theorem 6, 34, 12, 13, 41 and Lemma 7.

In the present section we introduce two more results of this type.

We make two remarks before formulating the first result. First of all, in the case $n=1$, $m=2$ (as pointed out in § 2 and § 5.3) there exist infinitely many consecutive triples of linearly independent best approximation vectors. Therefore, for infinitely many values of ν we have

$$\zeta_\nu \geqslant \frac{1}{6M_{\nu+1}M_{\nu+2}}\,.$$

Second, we note that in the author's paper [35] the following result was announced: for $n=1$ and $m\geqslant 2$ and for a given function $\psi(t)$ decreasing to zero (arbitrarily fast) as $t\to+\infty$, there exist vectors $\Theta\in\mathbb{R}^m$ such that $\dim_{\mathbb{Z}}\Theta=m+1$ and

$$\zeta_\nu \leqslant \psi(M_{\nu+m-1}).$$

There is a complete proof in [33].

This result can be essentially improved.

Theorem 68. *Let $m\geqslant 3$. Suppose that the function $\psi(t)$ decreases to zero (arbitrarily fast) as $t\to+\infty$. Let $\tau(\nu)$, $\nu=1,2,3,\dots$, be an increasing sequence of positive integers. Then there exist vectors $\Theta\in\mathbb{R}^m$ such that $\dim_{\mathbb{Z}}\Theta=m+1$ and*

$$\zeta_\nu \leqslant \psi(M_{\nu+\tau(\nu)}).$$

We give a sketched proof of the theorem in the case $m=3$.

For vectors $\mathbf{w},\mathbf{e}\in\mathbb{R}^3$ (here $\mathbf{e}$ is always a vector of unit length) and positive numbers η and δ we define a ball sector

$$B_{\eta,\delta}(\mathbf{w},\mathbf{e})=\left\{\Theta : |\Theta-\mathbf{w}|\leqslant\eta,\ \left|\frac{\Theta-\mathbf{w}}{|\Theta-\mathbf{w}|}-\mathbf{e}\right|\leqslant\delta\right\} \tag{107}$$

(here $|\cdot|$ stands for the Euclidean norm, but the type of norm is not important). For a vector $\mathbf{w}=(w^1,w^2,w^3)\in\mathbb{R}^3$ let $\overline{\mathbf{w}}=(w^1,w^2,w^3,1)\in\mathbb{R}^4$. We identify the space $\mathbb{R}^3$ with the affine subspace $\{(\mathbf{x},y): y=1\}\subset\mathbb{R}^4$. To prove Theorem 68 we need the following lemma.

Lemma 8. *Consider a three-dimensional completely rational linear subspace $\Pi\subset\mathbb{R}^4$, $\dim\Pi=3$, with normal vector $\overline{\mathbf{w}}$, and let $\pi_1,\pi_2\subset\Pi$ be two linear subspaces with $\dim\pi_1=\dim\pi_2=2$. Let $a<b<c$ be positive integers and let the vectors*

$$\mathbf{z}_\nu=(\mathbf{x}_\nu,y_\nu)=(x_{1,\nu},x_{2,\nu},x_{3,\nu},y_\nu)\in\mathbb{Z}^4,\qquad 1\leqslant\nu\leqslant c,$$

be such that:

(i) $\mathbf{z}_a,\mathbf{z}_{a+1},\dots,\mathbf{z}_{b-1},\mathbf{z}_b\in\pi_1$;

(ii) $\mathbf{z}_b,\mathbf{z}_{b+1},\dots,\mathbf{z}_{c-1},\mathbf{z}_c\in\pi_2$;

(iii) $\mathbf{z}_{b-1},\mathbf{z}_b,\mathbf{z}_{b+1}$ *are linearly independent.*

For some positive numbers η and δ and for a vector $\mathbf{e} \in \mathbb{R}^3$ suppose that $\mathbf{e}$ is orthogonal to π_2, and that for any $\Theta \in B_{\eta,\delta}(\mathbf{w}, \mathbf{e})$ with $\dim_{\mathbb{Z}} \Theta = 4$ the first c best approximation vectors are just the vectors $\mathbf{z}_\nu$, $1 \leqslant \nu \leqslant c$.

Let $d > c$ be an integer.

Then there exist a completely rational linear subspace Π_* with $\dim \Pi_* = 3$ and with normal vector $\overline{\mathbf{w}}_*$, a subspace $\pi_3 \subset \Pi_*$ with $\dim \pi_3 = 2$, and a sequence of integer vectors

$$\mathbf{z}_{c+1}, \dots, \mathbf{z}_{d-1}, \mathbf{z}_d \in \pi_3$$

such that $\mathbf{z}_{c-1}$, $\mathbf{z}_c$, $\mathbf{z}_{c+1}$ are linearly independent, and there are positive numbers η_* and δ_* and a vector $\mathbf{e}_* \in \mathbb{R}^3$ such that $\mathbf{e}_*$ is orthogonal to π_3 and for any $\Theta \in B_{\eta_*,\delta_*}(\mathbf{w}_*, \mathbf{e}_*) \subset B_{\eta,\delta}(\mathbf{w}, \mathbf{e})$ with $\dim_{\mathbb{Z}} \Theta = 4$ the first d best approximation vectors are just the vectors $\mathbf{z}_\nu$, $1 \leqslant \nu \leqslant d$.

We now give a sketched proof of Lemma 8. The integer lattice $\mathbb{Z}^4$ is decomposed into three-dimensional affine sublattices Π_r (levels) parallel to the subspace Π:

$$\mathbb{Z}^4 = \bigcup_{r \in \mathbb{Z}} \Pi_r.$$

First of all we take an integer point $\mathbf{z}_{c+1}$ from the level Π_1 or Π_{-1} which is a 'neighbouring' level to $\Pi_0 = \Pi$ (the precise choice of this level depends on the direction of the vector $\mathbf{e}$). We can assume that the absolute value

$$M_{c+1} = \max_{i=1,2,3} |x_{i,c+1}|$$

is much bigger than all the parameters in the inductive assumption. Then define

$$\Pi_* = \operatorname{span}(\mathbf{z}_{c-1}, \mathbf{z}_c, \mathbf{z}_{c+1}), \qquad \pi_3 = \operatorname{span}(\mathbf{z}_c, \mathbf{z}_{c+1})$$

and let $\overline{\mathbf{w}}_*$ be the normal vector to Π_*.

The vectors $\mathbf{z}_c$, $\mathbf{z}_{c+1}$ lie in the two-dimensional completely rational subspace π_3. We can find a sequence of vectors $\mathbf{z}_{c+2}, \dots, \mathbf{z}_d \in \pi_3 \cap \mathbb{Z}^4$ that form the sequence of *all* best approximations (with norm greater than $|\mathbf{z}_{c+1}|_\bullet$) for the last vector $\mathbf{z}_d$ *with respect to the lattice* $\pi_3 \cap \mathbb{Z}^4$ with the induced norm. Since $\Pi_* \supset \pi_3 \ni \mathbf{z}_c, \dots, \mathbf{z}_d$, by taking small η_* and δ_* (dependent on $\rho(\Pi_*) > 0$ and $M_d = \max_{i=1,2,3} |x_{i,d}|$) we get the assertion of Lemma 8. In this procedure the direction of the vector $\mathbf{e}_*$ is chosen in such a way that the vectors $\mathbf{z}_{d-1}$ and $\mathbf{z}_d$ (but not the vectors $\mathbf{z}_d - \mathbf{z}_{d-1}$ and $\mathbf{z}_d$) are the last best approximations from the first d best approximations for any Θ in the constructed neighbourhood. (Such a construction was developed by German in [111], [112].)

To prove Theorem 68 we must use Lemma 8 in the following way. We first enumerate all three-dimensional completely rational subspaces of $\mathbb{R}^4$. Then we start an inductive process for constructing embedded neighbourhoods of the form (107). The sequence of indices ν_k is defined to satisfy $\nu_{k+1} = \nu_k + \tau(\nu_k)$. At each step Lemma 8 is used with $a = \nu_{k-1}$, $b = \nu_k$, $c = \nu_{k+1}$. In this process it is necessary in addition to choose η_* small enough that

$$\eta_* < \frac{\psi(M_d)}{8M_b}\,.$$

Then for $b \leqslant \nu \leqslant c$ we get that $\nu + \tau(\nu) \leqslant c + \tau(c) = d$ and

$$\zeta_\nu \leqslant \zeta_b < \psi(M_d) \leqslant \psi(M_{\nu+\tau(\nu)}).$$

Moreover, at each step we must avoid the completely rational subspace with the same number as the number of the step.

The second result we mention in this section was announced in [34].

Theorem 69. *Let $n = 1$ and $m \geqslant 2$. Then for any function $\psi(t)$ decreasing to zero (arbitrarily fast) as $t \to +\infty$ there exists a $\psi(t)$-singular $\Theta \in \mathbb{R}^m$ such that $\dim_{\mathbb{Z}} \Theta = m + 1$, and for any ν the vectors $\mathbf{z}_\nu, \mathbf{z}_{\nu+1}, \dots, \mathbf{z}_{\nu+m}$ are linearly independent.*

13. Appendix

13.1. Lacunary sequences. A sequence $\{t_j\}$, $j = 1, 2, 3, \dots$, of positive real numbers is said to be lacunary if for some $M > 0$

$$\frac{t_{j+1}}{t_j} \geqslant 1 + \frac{1}{M} \quad \forall\, j \in \mathbb{N}. \tag{108}$$

In 1926 Khintchine proved that for any sequence satisfying (108) there exist a real number α and a positive number γ such that

$$\|t_n \alpha\| \geqslant \gamma \quad \forall\, n \in \mathbb{N}.$$

This result was published in the paper [1] (Hilfssatz III). In the present survey we have referred to this paper many times. The recent book of selected works by Khintchine [16] also includes this paper. Lemma 2 in § 6.3 (p. 473) above is a natural generalization of this one-dimensional result of Khintchine.

Here we note that Khintchine's construction makes it possible to prove the existence of an absolute positive constant γ and a real number α such that

$$\|t_n \alpha\| \geqslant \frac{\gamma}{(M \log M)^2} \quad \forall\, n \in \mathbb{N}.$$

Half a century later, in 1975, Erdős [113] conjectured that for an arbitrary lacunary sequence there exists a real number α such that the fractional parts $\{\alpha t_j\}$, $j \in \mathbb{N}$, are not dense in $[0, 1]$. A resolution of Erdős' conjecture follows immediately from Khintchine's result cited above. But this result of Khintchine was forgotten. Other resolutions of the conjecture were published by Pollington [114] and de Mathan [115]. Quantitative improvements were due to Katznelson [116], Akhunzhanov and Moshchevitin [117], and Dubickas [118]. The best known result is due to Peres and Schlag [19], who proved that for some absolute positive constant $\gamma > 0$ and for any sequence $\{t_j\}$ under consideration there exists a real number α such that

$$\|\alpha t_j\| \geqslant \frac{\gamma}{M \log M} \quad \forall\, j \in \mathbb{N}. \tag{109}$$

They used an original construction involving a special version of Lovasz' local lemma.

This paper of Peres and Schlag was followed by several papers by others applying their method to various Diophantine problems (see [119]–[126]). For the problem on lacunary sequences the best value of the constant γ in (109) is apparently in the paper [125] by Rochev.

The Peres–Schlag method [19] for constructing badly approximable numbers (with respect to a lacunary sequence) has turned out to be useful in various problems in Diophantine approximation theory. We shall mention some such problems below, but first we make some comments concerning metrical results.

13.2. Some metrical results. First, we consider a one-dimensional approximation problem. We give a result of Cassels in [127]. Recall that an increasing sequence of integers $\{t_n\}_{n=1}^{\infty}$ is called a Σ-sequence if

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n\leqslant N} \frac{\mu_n}{t_n} > 0,$$

where μ_n denotes the number of fractions of the form j/t_n with $0 < j < t_n$ which are not of the form i/t_q with positive integers i and $q < n$. Examples of Σ-sequences are arbitrary lacunary sequences, the sequences n^d $(n = 1, 2, 3, \dots)$ with a fixed positive integer d, and the Furstenberg sequence (the definition will be given later).

Cassels' theorem in [127] asserts that if $\{t_n\}_{n=1}^{\infty}$ is a Σ-sequence and if $\psi(n)$ is a function decreasing to zero such that the series $\sum_{n=1}^{\infty} \psi(n)$ diverges, then for almost all ξ

$$\liminf_{n\to+\infty} \|t_n\xi\|(\psi(n))^{-1} = 0.$$

Second, we state the divergence case of Khintchine's theorem in [3] (for the problem of simultaneous approximations only, in the form as it appears in Chap. III of [24]). Suppose that a collection of positive-valued functions $\psi_1(q), \dots, \psi_n(q)$ of the natural number argument q is such that the function

$$\psi(q) = \prod_{j=1}^{n} \psi_j(q)$$

is non-increasing and the series $\sum_{q=1}^{+\infty} \psi(q)$ diverges. Then for almost all vectors $\Theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ there exist infinitely many integers q such that

$$\|q\theta_j\| < \psi_j(q), \qquad 1 \leqslant j \leqslant n.$$

Third, we refer to a result due to Gallagher [128] (here we give not the general result but a corollary) also dealing with the divergence case. This result states that for almost all pairs of real numbers (ξ_1, ξ_2)

$$\liminf_{n\to\infty} n \log^2 n \cdot \|n\xi_1\|\,\|n\xi_2\| = 0.$$

13.3. Applications of the Peres–Schlag method. In this subsection we discuss some problems where the Peres–Schlag method gives non-trivial results on the existence of badly approximable numbers.

A. Approximations with subexponential sequences. Consider a sequence of positive integers t_n, $n=1,2,3,\dots$, such that

$$t_n \asymp \exp(\gamma n^{\beta}), \qquad \gamma>0, \quad 0<\beta<1. \tag{110}$$

In [121] it is shown that for any sequence of real numbers η_n the set

$$\Bigl\{\xi\in\mathbb{R} : \inf_{n\in\mathbb{N}} \|t_n\xi-\eta_n\|\cdot n^{1-\beta}\log n>0\Bigr\}$$

has full Hausdorff dimension 1. (More precisely, only the homogeneous case $\eta_n=0$ for any n was considered in [121], but the proof for the general case follows the proof for the particular case word for word.)

As an example we consider the Furstenberg sequence s_n, $n=1,2,3,\dots$, consisting of integers of the form $2^k\cdot 3^m$, $k,m=0,1,2,3,\dots$, arranged in increasing order. For this sequence the condition (110) is satisfied with $\beta=1/2$. Hence we see that for any fixed sequence η_n the set

$$\Bigl\{\xi\in\mathbb{R} : \inf_{n\in\mathbb{N}} \|s_n\xi-\eta_n\|\cdot n^{1-\beta}\log n>0\Bigr\}$$

has full Hausdorff dimension 1.

Furstenberg [129] (a simple proof is given by Boshernitzan [130]) proved that for an irrational number ξ the fractional parts $\{2^n\cdot 3^m\alpha\}$ are dense in $[0,1]$, and so

$$\liminf_{n\to\infty}\|s_n\xi\|=0.$$

There is a wonderful recent result due to Bourgain, Lindenstrauss, Michel, and Vencatesh about the rate of convergence to zero of $\|s_n\alpha\|$. It involves lower bounds for an integer linear form in two logarithms of algebraic numbers.

B. Polynomial-like sequences. Consider a polynomial-like increasing sequence t_n such that

$$t_n\asymp n^{\beta}, \qquad \beta>0.$$

In [122] it was proved that the set

$$\Bigl\{\xi\in\mathbb{R} : \inf_{n\in\mathbb{N}} \|t_n\xi-\eta_n\|\cdot n\log n>0\Bigr\}$$

has Hausdorff dimension $\geqslant \beta/(1+\beta)$.

This result gives a positive answer to a question of Schmidt in [132]. In the same paper he asks a more difficult question which is still open: does there exist a real number ξ such that

$$\inf_{n\in\mathbb{N}}\|n^2\xi\|n>0\,?$$

Here we cite a result of Zaharescu [133]: for any irrational number ξ and any positive ε

$$\liminf_{n\to\infty}\|n^2\xi\|n^{2/3-\varepsilon}=0.$$

This result is the best known up to now.

Cassels' theorem cited in § 13.2 shows that the sets constructed in **A** and **B** above have zero Lebesgue measure. In [125] Rochev obtained a general theorem on approximations for linear forms that generalizes the results above on the existence of badly approximable numbers.

In the present subsection we now discuss two multidimensional problems.

C. The Littlewood problem. The famous Littlewood conjecture states that for any two real numbers ξ_1 and ξ_2

$$\liminf_{n\to\infty} n\,\|n\xi_1\|\,\|n\xi_2\| = 0.$$

By the way, the author would like to say the he became familiar with the Littlewood problem when he was a student. It was S. A. Dovbysh who first told the author about this conjecture long ago.

By the Peres–Schlag approach one can prove (see [124]) the existence of ξ_1 and ξ_2 such that

$$\liminf_{n\to\infty} n\log^2 n\cdot\|n\xi_1\|\,\|n\xi_2\| > 0.$$

From Gallagher's theorem cited at the end of § 13.2 it follows that the pairs ξ_1, ξ_2 with such a property form a set of zero Lebesgue measure in $\mathbb{R}^2$.

D. BAD-conjecture. For $\alpha,\beta\in[0,1]$ with $\alpha+\beta=1$ and for $\delta\in(0,1/2)$ we consider the set

$$\mathrm{BAD}(\alpha,\beta;\delta)=\Bigl\{\xi=(\xi_1,\xi_2)\in[0,1]^2:\inf_{p\in\mathbb{N}}\max\{p^\alpha\|p\xi_1\|,p^\beta\|p\xi_2\|\}\geqslant\delta\Bigr\}.$$

Schmidt conjectured in [63] that for any two pairs $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in[0,1]^2$ with $\alpha_1+\beta_1=\alpha_2+\beta_2=1$ the intersection

$$\mathrm{BAD}(\alpha_1,\beta_1)\cap\mathrm{BAD}(\alpha_2,\beta_2)$$

is non-empty. This conjecture is dual to the Littlewood conjecture, and is now known as the BAD-conjecture. It remained unsolved for a long time. Recently Badziahin, Pollington, and Velani in the wonderful paper [134] obtained an elegant solution of this problem. What is more, they proved that the intersection of any finite (and under some additional assumptions countable) family of sets of the form $\mathrm{BAD}(\alpha_k,\beta_k)$ has full Hausdorff dimension.

The Peres–Schlag method gives the following result. For any sequence of inhomogeneities $\eta=\{\eta_j\}_{j=1}^\infty$ consider the set

$$\begin{aligned}\mathrm{BAD}^*_\eta(\alpha,\beta;\delta)=\Bigl\{&\xi=(\xi_1,\xi_2)\in[0,1]^2:\\ &\inf_{p\in\mathbb{N}}\max\{(p\log(p+1))^\alpha\|p\xi_1\|,\ (p\log(p+1))^\beta\|p\xi_2-\eta_p\|\}\geqslant\delta\Bigr\}.\end{aligned}$$

It is clear that

$$\mathrm{BAD}(\alpha,\beta;\delta)\subseteq\mathrm{BAD}^*_0(\alpha,\beta;\delta)$$

(here $\eta=0$ means that all the η_j are equal to zero) and that since the series

$$\sum_{p=1}^\infty\frac{1}{p\log(p+1)}$$

diverges, for any sequence η the union

$$\mathrm{BAD}^*_\eta(\alpha,\beta) = \bigcup_{\delta>0} \mathrm{BAD}^*_\eta(\alpha,\beta;\delta)$$

is a set of zero Lebesgue measure.

In [123] it is proved that for $\alpha_1,\alpha_2,\beta_1,\beta_2 \in [0,1]$ with $\alpha_1+\beta_1=\alpha_2+\beta_2=1$ and for $0<\delta\leqslant 2^{-20}$ the intersection

$$\mathrm{BAD}^*_0(\alpha_1,\beta_1;\delta) \cap \mathrm{BAD}^*_0(\alpha_2,\beta_2;\delta) \tag{111}$$

is non-empty. The proof of the fact that the sets

$$\mathrm{BAD}^*_{\eta^1}(\alpha_1,\beta_1;\delta) \cap \mathrm{BAD}^*_{\eta^2}(\alpha_2,\beta_2;\delta)$$

are non-empty for any fixed sequences η^1 and η^2 follows the proof of (111) in [123] word for word.

Bugeaud showed in the joint paper [126] that in all the applications of the Peres–Schlag method which were considered above, the method constructs sets of *full Hausdorff dimension*. To get such a result one should use the mass distribution principle (see [135], Chap. V, or [136]). Moreover, in the paper [126] there are some other applications of that method to Littlewood-type problems.

We do not presume to give here a detailed survey of papers and results related to Littlewood-type problems. There are many such papers. We note only that there are a large number of them dealing with the problems discussed above by means of dynamical systems. We refer to the survey [58].

The proofs of all the results in this subsection relating to the Peres–Schlag method are based on a certain application of a special form of Lovasz' local lemma (see [125]).

Lemma 9. *Suppose that* $\{A_n\}_{n=1}^N$ *is a system of events in a probability space* $(\Omega,\mathscr{F},\mathbf{P})$. *Let* $\{x_n\}_{n=1}^N$ *be a collection of numbers in* $[0,1]$. *Put* $B_0=\Omega$ *and* $B_n=\bigcap_{m=1}^n A_m^c$ $(1\leqslant n\leqslant N)$, *where* $A_m^c=\Omega\setminus A_m$. *Suppose that for any* $n\in\{1,\dots,N\}$ *there exists an* $m=m(n)\in\{0,1,\dots,n-1\}$ *such that*

$$\mathbf{P}(A_n\cap B_m) \leqslant x_n \prod_{m<k<n}(1-x_k)\cdot \mathbf{P}(B_m)$$

(*if* $m=n-1$, *take* $\prod_{m<k<n}(1-x_k)=1$). *Then for* $1\leqslant n\leqslant N$

$$\mathbf{P}(B_n) \geqslant (1-x_n)\mathbf{P}(B_{n-1}).$$

13.4. $(\boldsymbol{\alpha},\boldsymbol{\beta})$-games. We recall the definition of a Mazur–Banach–Schmidt (α,β)-game in the space $\mathbb{R}^d$. Let $\alpha,\beta\in(0,1)$ and let $S\subseteq\mathbb{R}^d$ be a set. The two players are White and Black. First, Black picks a closed ball (in the sup-norm) B_1 of diameter $l(B_1)=2\rho$. Then White picks a closed ball $W_1\subset B_1$ of diameter $l(W_1)=\alpha l(B_1)$. Then Black picks a closed ball $B_2\subset W_1$ of diameter $l(B_2)=\beta l(W_1)$, and so on. As a result, we have a sequence of nested closed balls $B_1\supset W_1\supset B_2\supset W_2\supset\cdots$ with diameters $l(B_i)=2(\alpha\beta)^{i-1}\rho$ and $l(W_i)=2\alpha(\alpha\beta)^{i-1}\rho$ $(i=1,2,\dots)$. Obviously, the set $\bigcap_{i=1}^\infty B_i=\bigcap_{i=1}^\infty W_i$ consists of just one point. If $\bigcap_{i=1}^\infty B_i\in S$, then we say that White has won the game. A set S is said to be (α,β)-winning if White can win independently of how Black plays. A set S is said to be α-winning if it is (α,β)-winning for all $\beta\in(0,1)$. The winning dimension $\operatorname{windim}\mathscr{A}$ of a set $\mathscr{A}\subseteq\mathbb{R}^d$ is defined as the supremum of those α for which the set $\mathscr{A}$ is α-winning. In the case $\operatorname{windim}\mathscr{A}>1/2$ we obviously have $\mathscr{A}=\mathbb{R}^d$. For $0<\alpha\leqslant 1/2$ there exist non-trivial α-winning sets. Schmidt [20], [24] proved a series of general theorems about winning sets. Two of them are stated below.

Theorem 70. *Let $\alpha>0$. Then any α-winning set in $\mathbb{R}^d$ has Hausdorff dimension d.*

Theorem 71. *Let $\alpha>0$. Then the intersection of a countable collection of α-winning sets is again an α-winning set.*

Here are some examples of winning sets.

A. The following example is due to Schmidt [20].

Theorem 72. *Let $d=1$, and fix any integer $q\geqslant 2$. The set*

$$\mathscr{N}_q=\{x\in\mathbb{R}:\ \exists C(x)>0 \text{ with } \|q^nx\|\geqslant C(x)\ \forall\, n\in\mathbb{N}\}$$

of numbers which are not normal with respect to the natural number base q has winning dimension $1/2$.

B. For $d=1$ consider a lacunary sequence of positive numbers t_j (the definition is in § 13.1). Generalizing Schmidt's arguments, we easily see that the set

$$\mathscr{N}=\{x\in\mathbb{R}:\exists\, C(x)>0 \text{ with } \|t_nx\|\geqslant C(x)\ \forall\, n\in\mathbb{N}\}$$

has winning dimension $1/2$.

C. For sequences with a sublacunary rate of growth the author proved the following result in [137].

Theorem 73. *Suppose that a sequence of numbers t_n is such that*

$$\forall\,\varepsilon>0\ \exists N_0\ \forall\, n\geqslant N_0:\ \frac{t_{n+1}}{t_n}\geqslant 1+\frac{1}{n^{\varepsilon}}\,.$$

Then for any $\delta>0$ the set

$$\mathscr{A}_\delta=\{x\in\mathbb{R}:\ \exists c(x)>0 \text{ with } \|t_nx\|>c(x)/n^{\delta}\ \forall\, n\in\mathbb{N}\}$$

is an α-winning set for any $\alpha\in(0,1/2)$, and hence $\operatorname{windim}\mathscr{A}_\delta=1/2$.

D. We should say a few words about badly approximable systems of linear forms in particular. A collection $\Theta=\{\theta^i_j,\ 1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n\}$ of real numbers is said to be *badly approximable* if there exists a positive number γ such that for the corresponding system $\mathbf{L}_\Theta(\mathbf{x})=\{L_j(\mathbf{x}),\ 1\leqslant j\leqslant n\}$ of linear forms

$$\max_{1\leqslant j\leqslant n}\|L_j(\mathbf{x})\|\geqslant\gamma\cdot\Bigl(\max_{1\leqslant i\leqslant m}|x_i|\Bigr)^{-m/n}\qquad\forall\,\mathbf{x}\in\mathbb{Z}^m\setminus\{\mathbf{0}\}.$$

It is a well-known fact that badly approximable systems Θ exist and that for any n, m the set of all badly approximable systems Θ has zero Lebesgue measure in $\mathbb{R}^{nm}$ (for example, this follows from Khintchine's metrical theorem cited in § 13.2).

On the other hand, Θ is badly approximable if and only if the Jarník function (20) satisfies

$$\liminf_{t\to+\infty} t^{m/n}\psi_\Theta(t) > 0.$$

Of course, a badly approximable system Θ forms a regular system (in the sense of Khintchine's terminology).

We state Schmidt's theorem in [138].

Theorem 74. *For any dimensions n and m the set of all badly approximable systems Θ in $\mathbb{R}^d$ with $d = nm$ has winning dimension* $1/2$.

There is an enormous amount of literature devoted to badly approximable systems, and the author has simply not been able to put together a good survey on this topic.

E. In concluding, we note that there are fairly many different papers devoted to winning sets (see for example [51], [52], [56], [139]–[145]).

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