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Symmetry reductions of Lax integrable 3D systems

(joint work with H. Baran, O. Morozov, and P. Vojčák)

We give a complete description of symmetry reductions for the following 3D Lax integrable (i.e., admitting a ZCR with a non-removable parameter) equations:

the Pavlov equation	$u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}$,	(1)
the 3D rdDym equation	$u_{ty} = u_x u_{xy} - u_y u_{xx}$,	(2)
the universal hierarchy equation	$u_{yy} = u_z u_{xy} - u_y u_{xz}$	(3)

(see [1] and references therein). The result comprised more than 30 equations, but the majority of them were either exactly solvable or linearized by the generalized Legendre transformations. Nevertheless, there were 10 'interesting' reductions, among which two well known equations, i.e., the Liouville[1] and Gibbons-Tsarev equations. The rest nine can be divided in two groups by their symmetry properties: five equations admit infinite-dimensional Lie algebras of contact symmetries (with functional parameters) and four others possess finite-dimensional symmetry algebras. The integrability properties of these four equations were studied in [2] and the main results are as follows.

Equation (1) admits the covering

$$q_t = (q^2 - qu_x - u_y)q_x, \qquad q_y = (q - u_x)q_x.$$

The symmetry $\varphi_1 = u_t - 2xu_x - yu_y + 3u$ lifts to this covering and the reduction leads to the equation

$$v_{\eta\eta} = (v_\eta + 2\xi)v_{\xi\xi} - (v_\xi - \eta)v_{\xi\eta} - v_\xi \tag{4}$$

and the covering

$$w_\xi = \frac{-w}{w^2 - (v_\xi + \eta)w + \eta v_\xi - v_\eta - 2\xi}, \quad w_\eta = \frac{-w(w - v_\xi)}{w^2 - (v_\xi + \eta)w + \eta v_\xi - v_\eta - 2\xi}.$$

The reduction with respect to the symmetry $\varphi_2 = u_t - yu_x + 2x$ leads to the equation

$$v_{\eta\eta} = (v_\eta + \eta)v_{\xi\xi} - v_\xi v_{\xi\eta} - 2 \tag{5}$$

with the covering

$$w_\xi = \frac{-1}{w^2 - v_\xi w - v_\eta - \eta}, \quad w_\eta = \frac{v_\xi - w}{w^2 - v_\xi w - v_\eta - \eta}$$

By the change of variable $v \mapsto v - \eta^2/2$ Equation (5) reduces to the Gibbons-Tsarev equation, while the covering becomes the well known nonlinear Lax pair of this equation.

Equation (2) admits the covering

$$q_t = (u_x + q)q_x, \qquad q_y = -\frac{u_y q_x}{q}.$$

[1] Which is also linearizable by a well known differential substitution and is not considered below.

The symmetry $\varphi = u_t - xu_x - u_y + 2u$ can be prolonged to a symmetry of the covering and as the result of φ-reduction we obtain the equation

$$v_{\eta\eta} = (v_\xi - \xi)v_{\xi\eta} - v_\eta(v_{\xi\xi} - 2) \tag{6}$$

with the covering

$$w_\xi = \frac{-w^2}{w^2 + (v_\xi - \xi)w + v_\eta}, \quad w_\eta = \frac{v_\eta w}{w^2 + (v_\xi - \xi)w + v_\eta}.$$

Finally, Equation (3) admits the covering

$$q_z = \frac{(qu_z - u_y)q_x}{q^2}, \qquad q_y = \frac{u_y q_x}{q}$$

and the reduction with respect to the symmetry $\varphi = u_z + u_x + yu_y + u$ prolonged to the covering leads to the equation

$$v_{\eta\eta} = v_\eta v_{\xi\xi} - (v_\xi + v)v_{\xi\eta} + v_\xi v_\psi \tag{7}$$

with the covering

$$w_\xi = \frac{-w^3}{w^2 - (v_\xi + v)w - v_\eta}, \quad w_\eta = \frac{-v_\eta w^2}{w^2 - (v_\xi + v)w - v_\eta}.$$

Equations (4)–(6) are pair-wise inequivalent with respect to contact transformation.

Using the standard reversal procedure, i.e., passing from a one-dimensional covering

$$w_\xi = X(\xi, \eta, v, v_\xi, v_\eta, w), \qquad w_\eta = Y(\xi, \eta, v, v_\xi, v_\eta, w)$$

to the infinite-dimensional covering

$$\psi_\xi = -X(\xi, \eta, v, v_\xi, v_\eta, \lambda)\psi_\lambda, \qquad \psi_\eta = -Y(\xi, \eta, v, v_\xi, v_\eta, \lambda)\psi_\lambda, \tag{8}$$

and expanding (8) in formal Laurent series in λ, we constructed infinite hierarchies of nonlocal conservation laws for Equations (4)–(6).

References

[1] H. Baran, I.S. Krasil′shchik, O.I. Morozov, P. Vojčák. Symmetry reductions and exact solutions of Lax integrable 3-dimensional systems, J. of Nonlinear Math. Phys., **21** (2014) No. 4, 643–671; arXiv:1407.0246 [nlin.SI].

[2] H. Baran, I.S. Krasil′shchik, O.I. Morozov, P. Vojčák. Integrability properties of some equations obtained by symmetry reductions, J. of Nonlinear Math. Phys., **22** (2015) No. 2, 210–232; arXiv:1412.6461 [nlin.SI].