Lecture 10

THE FUNDAMENTAL GROUP

The fundamental group is one of the most important invariants of homotopy theory. It also has numerous applications outside of topology, especially in complex analysis, algebra, theoretical mechanics, and mathematical physics. In our course, it will be the first example of a "functor", assigning a group to each pathconnected topological space and a group homomorphism to each continuous map of such spaces, thus reducing topological problems about spaces to problems about groups, which can often be effectively solved.

10.1. Main definitions

Let M be a topological space with a distinguished point $p \in M$. A curve $c\colon [0,1] \to M$ such that $c(0) = c(1) = p$ will be called a *loop* with *basepoint* p. Two loops c_0, c_1 with basepoint p are called *homotopic rel endpoints* if there is a homotopy $F : X \times [0,1] \to Y$ joining c_0 to c_1 such that $F(t,x) = p$ for all $t \in [0,1]$.

Two curves c_0, c_1 such that $c_0(x) = c_1(x) = p$ (not necessarily loops) are called *homotopic rel* p if there is a homotopy H joining c_0 to c_1 such that $H(t,x) = p$ for all $t \in [0,1]$.

If c_1 and c_2 are two loops with basepoint p, then the loop $c_1 \cdot c_2$ given by

$$c_1 \cdot c_2(t) := \begin{cases} c_1(2t) & \text{if } t \le \frac{1}{2}, \\ c_2(2t-1) & \text{if } t \ge \frac{1}{2}. \end{cases}$$

is called the *product* of c_0 and c_1.

Proposition 10.1. *Classes of loops homotopic rel endpoints form a group with respect to the product operation induced by* $\cdot$.

Proof. First notice that the operation is indeed well defined on the homotopy classes. For, if the paths c_i are homotopic to $\tilde{c}_i,\ \ i = 1,2$ via the maps $h_1 : [0,1] \times [0,1] \to M$, then the map h, defined by

$$h(t,s) := \begin{cases} h_1(2t,s) & \text{if } \ t \le \frac{1}{2}, \\ h_2(2t-1,s) & \text{if } t \ge \frac{1}{2} \end{cases}$$

is a homotopy rel endpoints joining c_1 to c_2.

Obviously, the role of the unit is played by the homotopy class of the constant map $c_0(t) = p$. Then the inverse to c will be the homotopy class of the map $c'(t) := c(1-t)$. What remains is to check the associative law: $(c_1 \cdot c_2) \cdot c_3$ is homotopic rel p to $c_1 \cdot (c_2) \cdot c_3)$ and to show that $c \cdot c'$ is homotopic to c_0. In both cases the homotpy is done by a reparametrization in the preimage, i.e., on the square $[0,1] \times [0,1]$.

For associativity, consider the following continuous map ("reparametrization") of the square into itself

$$R(t,s) = \begin{cases} (t(1+s), s) & \text{if } 0 \le t \le \frac{1}{4}, \\ (t + \frac{s}{4}, s) & \text{if } \frac{1}{4} \le t \le \frac{1}{2}, \\ (1 - \frac{1}{1+s} + \frac{t}{1+s}, s) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then the map $c_1 \cdot (c_2 \cdot c_3) \circ R : [0,1] \times [0,1] \to M$ provides a homotopy rel endpoints joining the loops $c_1 \cdot (c_2 \cdot c_3)$ and $(c_1 \cdot c_2) \cdot c_3$.

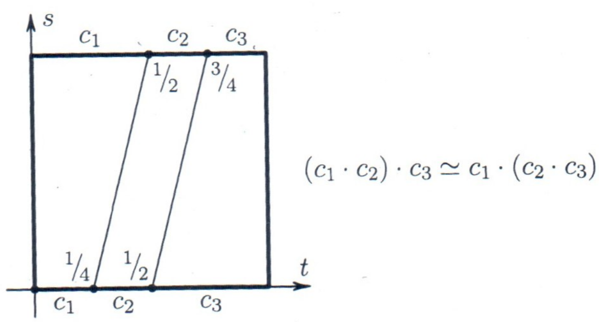

FIGURE 10.1. Associativity of multiplication

Similarly, a homotopy joining $c \cdot c'$ to c_0 is given by $c \cdot c' \circ I$, where the reparametrization $I : [0,1] \times [0,1] \to [0,1] \times [0,1]$ is defined as

$$I(t,s) = \begin{cases} (t, s) & \text{if } 0 \le t \le \frac{1-s}{2}, \text{ or } \frac{1+s}{2} \le t \le 1, \\ (\frac{1-s}{2}, s) & \text{if } \frac{1-s}{2} \le t \le \frac{1+s}{2}, \end{cases}$$

Notice that while the reparametrization I is discontinuous along the wedge $t = (1 \pm s)/2$, the map $(c \cdot c') \circ I$ is continuous by the definition of c'. $\square$

The group described in Proposition 10.1 is called the *fundamental group* of M at p and is denoted by $\pi_1(M,p)$.

It is natural to ask to what extent $\pi_1(M,p)$ depends on the choice of the point $p \in M$. The answer is given by the following proposition.

Proposition 10.2. *If p and q belong to the same path connected component of M, then the groups $\pi_1(M,p)$ and $\pi_1(M,q)$ are isomorphic.*

Proof. Let $\rho : [0,1] \to M$ be a path connecting the points p and q. It is natural to denote the path $\rho \circ S$ where $S(t) = 1-t$ by ρ^{-1}. It is also natural to extend the "$\cdot$" operation to paths with different endpoints if they match properly. With these conventions established, let us associate to a path c: $[0,1] \to M$ with $c(0) = c(1) = p$ the path $c'\rho^{-1} \cdot c \cdot \rho$ with $c'(0) = c'(1) = q$. In order to finish the proof, we must show that this correspondence takes paths homotopic rel p to paths homotopic rel$q$, respects the group operation and is bijective up to homotopy. These statements are proved using appropriate rather natural reparametrizations, as in the proof of Proposition 10.1.

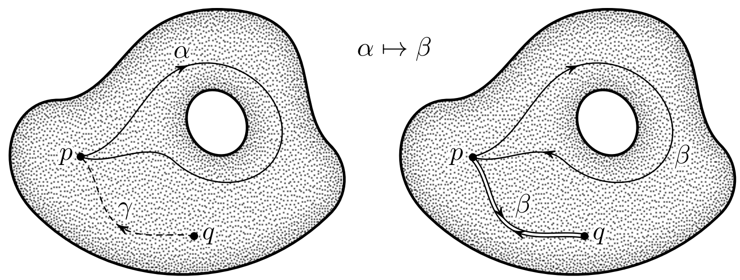

FIGURE 10.2. Change of basepoint isomorphism

Remark 10.1. It follows from the construction that different choices of the connecting path ρ will produce isomorphisms between $\pi_1(M,p)$ and $\pi_1(M,q)$ which differ by an inner automorphism of either group.

If the space M is path connected, then the fundamental groups at all of its points are isomorphic and one simply talks about the *fundamental group of* M and often omits the basepoint from its notation: $\pi_1(M)$.

The *free* homotopy classes of curves (i.e., with no fixed base point) correspond exactly to the conjugacy classes of curves modulo changing base point, so there is a natural bijection between the classes of freely homotopic closed curves and conjugacy classes in the fundamental group.

A path connected space with trivial fundamental group is said to be *simply connected* (or sometimes 1-*connected*).

Remark 10.2. Since the fundamental group is defined modulo homotopy, it is the same group for homotopically equivalent spaces, i.e., *the fundamental group* $\pi_1(M)$ *is a homotopy invariant.*

10.2. Functoriality

Now suppose that X and Y are path connected, $f : X \to Y$ is a continuous maps with and $f(p) = q$. Let $[c]$ be an element of $\pi_1(X,p)$, i.e., the homotopy class rel endpoints of some loop $c : [0,1] \to X$. Denote by $f_\#(c)$ the loop in (Y,q) defined by $f_\#(t) := f(c(t))$ for all $t \in [0,1]$.

Proposition 10.3. *The assignment* $c \mapsto f_\#(c)$ *is well defined on classes of loops and determines a homomorphism (still denoted by* $f_\#$*) of fundamental groups:*

$$f_\# : \pi_1(X,p) \to \pi_1(Y,q)$$

(called the homomorphism induced by f *), which possesses the following properties (called functorial):*

- $(f \circ g)_\# = f_\# \circ g_\#$ *(covariance);*
- $(id_X)_\# = id_{\pi_1(X,p)}$ *(identity maps induce identity homomorphisms).*

The fact that the construction of an invariant (here the fundamental group) is functorial is very convenient for applications, as seen in the following example.

Example 10.1. Let us give another proof of the Brouwer fixed point theorem for the disk by using the isomorphisms $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ and $\pi_1(\mathbb{D}^2) = 0$ (which will be established later) and the functoriality of $\pi_1(\cdot)$.

We will prove (by contradiction) that there is no retraction of $\mathbb{D}^2$ on its boundary $\mathbb{S}^1 = \partial\mathbb{D}^2$. Let $r : \mathbb{D}^2 \to \mathbb{S}^1$ be such a retraction, let $i : \mathbb{S}^1 \to \mathbb{S}^2$ be the inclusion; choose a basepoint $x_0 \in \mathbb{S}^1 \subset \mathbb{S}^2$. Note that for this choice of basepoint we have $i(x_0) = r(x_0) = x_0$. Consider the sequence of induced maps:

$$\pi_1(\mathbb{S}^1, x_0) \xrightarrow{i_*} \pi_1(\mathbb{D}^2, x_0) \xrightarrow{r_*} \pi_1(\mathbb{S}^1, x_0).$$

In view of the isomorphisms noted above, this sequence is actually

$$\mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}.$$

But such a sequence is impossible, because by functoriality we have

$$r_* \circ i_* = (r \circ i)_* = \mathrm{Id}_* = \mathrm{Id}_{\mathbb{Z}}\,.$$

The fundamental group behaves nicely with respect to Cartesian products, as the following proposition shows.

Proposition 10.4. *If X and Y are path connected spaces, then*

$$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y).$$

Proof. Let us construct an isomorphism of $\pi_1(X) \times \pi_1(Y)$ onto $\pi_1(X \times Y$. Let x_0, y_0 be the basepoints in X and Y, respectively. For the basepoint in $X \times Y$, let us take the point (x_0, y_0). Now to the pair of loops α and β in X and Y let us assign the loop $\alpha \times \beta$ given by $\alpha \times \beta(t) := (\alpha(t), \beta(t))$. The verification of the fact that this assignment determines a well-defined isomorphism of the appropriate fundamental groups is quite straightforward. For example, to prove surjectivity, for a given loop γ in $X \times Y$ with basepoint (x_0, y_0), we consider the two loops $\alpha(t) := (\mathrm{pr}_X \circ \gamma)(t)$ and $\beta(t) := (\mathrm{pr}_Y \circ \gamma)(t)$, where pr_X and pr_Y are the projections on the two factors of $X \times Y$.

Corollary 10.1. *If C is contractible, then $\pi_1(X \times C) = \pi_1(X)$.*

The proof is an exercise.

10.3. The Seifert–van Kampen theorem

In this section, we state without proof a classical theorem which relates the fundamental group of the union of two spaces with the fundamental groups of the summands and of their intersection. The result turns out to give an efficient method for computing the fundamental group of a “complicated"space by putting it together from “simpler"pieces.

In order to state the theorem, we need a purely algebraic notion from group theory.

Let G_i, $i = 1, 2$, be groups, and let $\varphi_i : K \to G_i$, $i = 1, 2$ be monomorphisms. Then the *free product with amalgamation* of G_1 and G_2 with respect to φ_1 and φ_2, denoted by

$G_1 *_K G_2$ is the quotient group of the free product $G_1 * G_2$ by the normal subgroup generated by all elements of the form $\varphi_1(k)(\varphi_2(k))^{-1}$, $k \in K$.

Theorem 10.1. (Van Kampen's Theorem) *Let the path connected space X be the union of two path connected spaces A and B with path connected intersection containing the basepoint $x_0 \in X$. Let the inclusion homomorphisms*

$$\varphi_A : \pi_1(A \cap B) \to \pi_1(A), \quad \varphi_A : \pi_1(A \cap B) \to \pi_1(B)$$

be injective. Then $\pi_1(X, x_0)$ is the amalgamated product

$$\boxed{\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)}$$

10.4. Exercises

10.1. Prove that if C is contractible, then $\pi_1(C) = 0$.

10.2. Prove that for any path connected topological space X we have $\pi_1(\mathrm{Cone(X)}) = 0$.

10.3. Prove that the fundamental group of the wedge product of n circles is isomorphic to the free group with n generators.

10.4. Prove that the group $\pi_1(n\mathbb{T}^2)$ is generated by elements a_1, b_1, …, a_n, b_n obeying to the unique relation

$$\prod_{i=1}^{n}(a_i b_i a_i^{-1} b_i^{-1}) = 1.$$

10.5. Prove that the group $\pi_1(n\mathbb{R}P^2)$ is generated by elements a_1, …, a_n, obeying to the unique relation $a_1^2 \dots a_n^2 = 1$.

10.6. (a) Prove that if $G = \pi_1(n\mathbb{T}^2)$, then $G/G' \cong \mathbb{Z}^{2n}$. (Here G' is the *commutant*, i.e. G' is the subgroup generated by all elements of the form $aba^{-1}b^{-1}$ for a, $b \in G$.)
(b) Prove that if $G = \pi_1(n\mathbb{R}P^2)$, then $G/G' \cong \mathbb{Z}^{n-1} \oplus \mathbb{Z}_2$.

10.7. Prove that $\pi_1(\mathbb{S}^n) = 0$ for $n \geq 2$.

10.8. Prove that $\pi_1(\mathbb{C}P^n) = 0$.

10.9. Prove that the fundamental group of the surface $n\mathbb{T}^2$ with $k \geq 1$ deleted discs is the free group of rank $2n + k - 1$.

10.10. Prove that the fundamental group of the surface $n\mathbb{R}P^2$ with $k \geq 1$ deleted discs is the free group of rank $n + k - 1$.

10.11. Suppose that X is the Möbius band, A is its boundary. Prove that A is not a retract of X.

10.12. Prove that any finite and connected CW-space is homotopy equivalent to CWcomplex with only one vertex e^0.