Lecture 6

HOMOTOPY

The notions of homotopy and homotopy equivalence are quite fundamental in topology. Homotopy equivalence of topological spaces is a weaker equivalence relation than homeomorphism, and *homotopy theory* studies topological spaces up to this relation (and maps up to homotopy). This theory constitutes the main body of *algebraic topology*, but we only consider a few of its basic notions here. One of these notions is the Euler characteristic, which is also a homotopy invariant.

6.1. Homotopic Maps

Two maps $f, g\colon X \to Y$ are called *homotopic* (notation $f \simeq g$) if they can be joined by a *homotopy*, i.e., by a map $F\colon X \times [0,1] \to Y$ such that $F(x,0) \equiv f(x)$ and $F(x,1) \equiv g(x)$ (here $\equiv$ means for all $x \in X$). If we change the notation from $F(x,t)$ to $F_t(x)$, we can restate the previous definition by saying that there exists a family $\{F_t(x)\}$ of maps, parametrized by $t \in [0,1]$, continiously changing from $f \equiv F_0$ to $g \equiv F_1$.

It is easy to prove that

$f \simeq f$ for any $f\colon X \to Y$ (*reflexivity*);

$f \simeq g \Longrightarrow f \simeq g$ for all $f, g\colon X \to Y$ (*symmetry*);

$f \simeq g$ and $g \simeq h \Longrightarrow f \simeq h$ for all $f, g, h\colon X \to Y$ (*transitivity*).

For example, to prove transitivity, we obtain a homotopy joining f and h by setting

$$F(x,t) = \begin{cases} F_1(x, 2t) & \text{for } 0 \le t \le 1/2, \\ F_2(x, 2t-1) & \text{for } 1/2 \le t \le 1, \end{cases}$$

where F_1, F_2 are homotopies joining f and g, g and h, respectively.

Thus the homotopy of maps is an equivalence relation, so that the set $\mathrm{Map}(X,Y)$ of all (continuous) maps of X to Y splits into equivalence classes, called *homotopy classes*; the set of these equivalence classes is denoted $[X,Y]$.

6.2. Homotopy Equivalence of Spaces

Two spaces X and Y are called *homotopy equivalent* if there exist two maps $f\colon X \to Y$, $g\colon Y \to X$ (called *homotopy equivalences*) such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Obviously, *homeomorphic spaces are homotopy equivalent* (the homotopy equivalences are provided by any homeomorphism and its inverse). The converse statement is not true: for example, the point is homotopy equivalent to the 2-disk, but these two spaces are not homeomorphic.

Thus homotopy equivalence is a weaker equivalence relation than homeomorphism, so that homotopy classification is rougher (and hence easier – there are less classes) than topological classification. Its importance in topology is due to the fact that most topological invariants are homotopy invariants (this is the case of the so-called fundamental group, homology groups and related invariants such as the Euler characteristic).

6.3. Degree of Maps of $\mathbb{S}^1$ into Itself

In this section we consider (continuous) maps $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ of the circle into itself. Examples are the maps $w_k\colon \mathbb{S}^1 \to \mathbb{S}^1$ given by the rule $e^{i\varphi} \mapsto e^{ik\varphi}$, where S^1 is modeled by unimodular complex numbers: $\mathbb{S}^1 = \{z \in \mathbb{C}\colon |z| = 1\}$.

Theorem 6.1. *There is a natural bijection between homotopy classes of maps of the circle into itself and the integers:*

$$\boxed{[\mathbb{S}^1, \mathbb{S}^1] \longleftrightarrow \mathbb{Z}.}$$

Proof. Consider the map $\exp\colon \mathbb{R} \to \mathbb{S}^1$ given by the rule $\mathbb{R} \ni \varphi \mapsto e^{i\varphi} \in \mathbb{S}^1$. The map exp is not a bijection; for example, it takes all points of the form $2k\pi$ to $1 \in \mathbb{S}^1$ (see Fig. 61 (a)). However, exp is a *local homeomorphism*, i.e., any point has a neighborhood U (e.g. any open interval of length less than 2π containing the point) such that the restriction $\exp|_U$ of exp to U is a homeomorphism.

Now any map $\mathbb{S}^1 \to \mathbb{S}^1$ can be regarded as a map $f\colon [0,1] \to \mathbb{S}^1$ such that $f(0) = f(1) = 1 \in \mathbb{S}^1$. For any such map there exists a unique map $\widetilde{f} : [0,1] \to \mathbb{S}$, called the *lift* of f, such that $\exp \circ \widetilde{f} = f$. Indeed, subdivide $[0,1]$ into segments $[0, a_1], [a_1, a_2], \dots, [a_m, 1]$, so small that none of the images of these segments covers $\mathbb{S}^1$; then, using the fact that exp is homeomorphic on each segment, we successively extend the map taking the point $0 \in [0,1]$ to the point $0 \in \mathbb{R}$ to a map $\widetilde{f}$ of the whole interval $[0,1]$ to $\mathbb{R}$. (Look at Figure 6.1(a).) We now define the required bijection by setting:

$$[\mathbb{S}^1, \mathbb{S}^1] \ni [f] \ni f \mapsto \widetilde{f}(1)/2\pi \in \mathbb{Z}.$$

To prove that this assignment is well defined (i.e., does not depend on the choice of f in $[f]$) and is bijective, it suffices to prove that *any map $f \in [f]$ is homotopic to some w_k.*

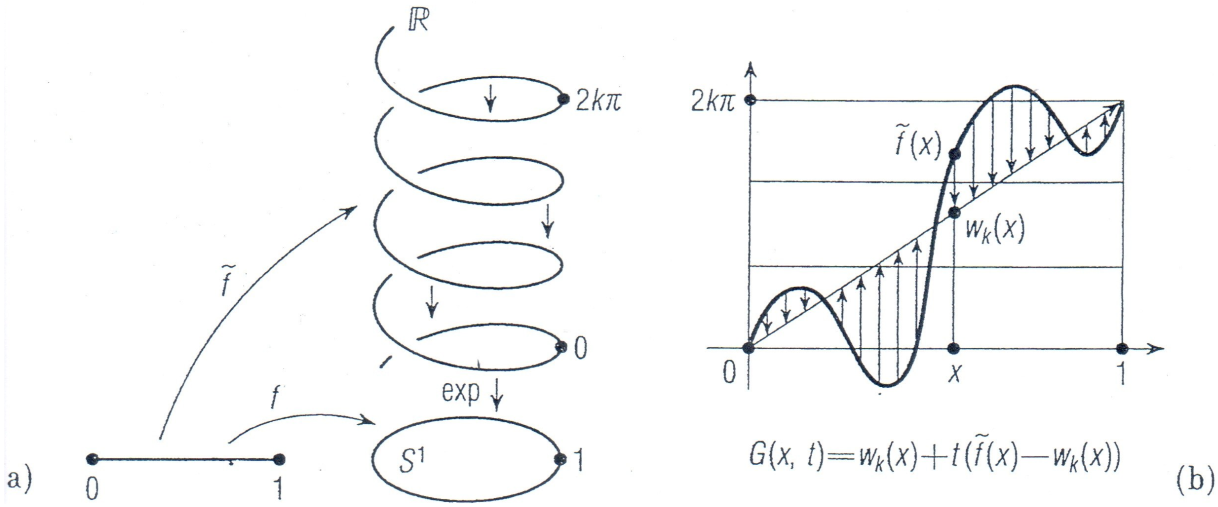

FIGURE 6.1. Liftings of the exponential map

Let $f \in [f] \in [\mathbb{S}^1, \mathbb{S}^1]$ be an arbitrary map, regarded as a map from $[0,1]$ to $\mathbb{S}^1$ such that $f(0) = f(1) = 1 \in \mathbb{S}^1$. Then the lift $\widetilde{f}$ is a (continuous) function defined on $[0,1]$ with values in $\mathbb{R}$ such that $\widetilde{f}(0) = 0 \in \mathbb{R}$. Let $\widetilde{f}(1) = 2k\pi$. The graph of $\widetilde{f}$ is shown on Figure 6.1 (b). The graph of w_k is a straight line joining the points $(0,0)$ and $(1, 2k\pi)$. There is an obvious homotopy (shown in Fig. 18 (b)) joining w_k and $\widetilde{f}$; denote it by $G(x,t)$. Then $F(x,t) := \exp(G(x,t))$ is the required homotopy between f and w_k.

The theorem is proved.

We can now define the *degree* of any circle map $f : \mathbb{S}^1 \to \mathbb{S}^1$ by setting

$$\boxed{\deg(f) := \widetilde{f}(1)/2\pi.}$$

The geometric meaning of the degree of a map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is "the number of times that the preimage circle wraps around the image circle. Thus the constant map $\mathbb{S}^1 \to 1 \in \mathbb{S}^1$ has degree 0 (the preimage circle wraps around the image circle zero times), the identity map has degree 1 (the preimage circle wraps around the image circle exactly once), the map w_{17} has degree 17 (the preimage circle wraps around the image circle seventeen times).

Corollary 6.2 *The identity map of the circle is not homotopic to the constant map* $\mathbb{S}^1 \to 1 \in \mathbb{S}^1$.

Remark. The notion of degree of a map can be generalized from maps of the circle to maps of the sphere $\mathbb{S}^n$ for any n, and even to arbitrary n-dimensional oriented manifolds. Although the definition is not difficult, it is hard to prove in the general case (i.e., for any $n \in \mathbb{Z}$) that the degree is well defined and depends only on the homotopy type of the map. To do that properly, you need *homology theory*, which lies outside the scope of this course.

6.4. A Fixed Point Theorem

The theorem proved in the previous section has numerous important corollaries, several of which be discussed in subsequent lectures. Here we only give one illustration, namely famous the Brouwer Fixed Point Theorem (for $n = 2$). Other more general fixed point theorems lie at the basis of fundamental existence theorems in differential equations and their applications to engineering and especially economics (the so-called *Nash equilibrium*), but they require homology theory for their proofs.

6.3. Brouwer Fixed Point Theorem. *Any continuous map of the* (*closed*) *disk has a fixed point, i.e., if* $f\colon \mathbb{D}^2 \to \mathbb{D}^2$ *is continuous, then there exists a point* $x \in \mathbb{D}^2$ *such that* $f(x) = x$.

For the proof, we will need a definition and a lemma. If A is a subspace of a topological space X, a continuous map $r\colon X \to A$ is said to be a *retraction* if r restricted to A is the identity. If a retraction $r\colon X \to A$ exists, then the subspace A is called a *retract* of X.

Lemma. *There is no retraction of the 2-disk on its boundary circle.*

Proof of the lemma. Suppose that there exists a retraction $r\colon \mathbb{D}^2 \to \partial\mathbb{D}^1$ of the 2-disk $\mathbb{D}^2$ on its boundary circle $\mathbb{S}^1 = \partial\mathbb{D}^2$. Consider the family $F_t(x)$ of maps $F_t\colon \mathbb{S}^1 \to \mathbb{S}^1$ given by the formula $F_t(e^{i\phi}) = r(te^{i\phi})$. The map F_0 is the constant map $\mathbb{S}^1 \to r(O)$ and the map F_1 (which is homotopic to F_0) is the identity map of the circle. This contradicts Corollary 6.2.

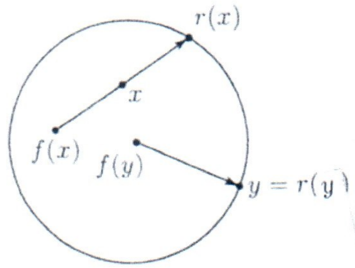

FIGURE 6.2. A retraction that does not exist

Proof of the theorem. To show that the Fixed Point Theorem follows from the lemma, assume that the theorem is false. For any $x \in D^2$, we have $f(x) \neq x$, and so the intersection point $r(x)$ of the ray $[f(x), x)$ with the boundary circle is well defined (look at Figure 6.2), and obviously the map $x \mapsto r(x)$ is a (continuous) retraction of $\mathbb{D}^2$ onto its boundary circle. But this contradicts the lemma. The theorem is proved.

6.5. Exercises

6.1. If the restrictions of a map $f : X \to Y$ to its closed subsets $X_1, \dots X_k$, where $X_1 \cup \dots \cup X_k = X$, are all continuous, then f is continuous.

6.2. (a) Prove that if a map $f : X \to \mathbb{S}^1$ is not surjective, then f is homotopic to the constant map.

(b) Prove that if a map $f : X \to \mathbb{S}^n$ is not surjective, then f is homotopic to the constant map.

6.3. Prove that the 2-sphere with two points identified and the union of the 2-sphere with one of its diameters are homotopy equivalent.

6.4. Prove that the spaces $\mathbb{S}^1 \wedge [0, 1]$ and $\mathbb{S}^1$ are homotopy equivalent. (Here and below $X \wedge Y$ denotes the *wedge* of the spaces X and Y connected spaces, i.e., the topological space obtained by identifying a point of X with a point of Y in the case when this topological space is well defined.)

6.5. Prove that the sphere with g handles from which a point has been removed is homotopy equivalent to the wedge of n circles and find n.

6.6. Prove that the spaces $\mathbb{S}^1 \wedge \mathbb{S}^2$ and $\mathbb{R}^3 \setminus \mathbb{S}^1$ are homotopy equivalent.

6.7. Let X be the space $\mathbb{R}^3$ from which k copies of the circle have been removed (the circles are unknotted and unlinked, i.e., they lie in nonintersecting balls). Prove that X is homotopy equivalent to the wedge product of k copies of the space $\mathbb{S}^1 \vee \mathbb{S}^2$.

6.8. Let L be the union of two circles in $\mathbb{R}^3$ linked in the simplest way. Prove that $\mathbb{R}^3 \setminus L$ is homotopy equivalent to the wedge $\mathbb{S}^2 \vee \mathbb{T}^2$.

6.9. Prove that the following assertions are equivalent:

(1) any continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point;

(2) there is no retraction $r : \mathbb{D}^n \to \partial\mathbb{D}^n$;

(3) for any vector field v on $\mathbb{D}^n$ such that $v(x) = x$ for all $x \in \partial\mathbb{D}^n$, there exists a point $x \in \mathbb{D}^n$ such that $v(x) = 0$ (for $n = 2$ this assertion is called "теорема о макушки" in Russian and "hedgehog theorem" in English).

6.10. Prove that A is a retract of X if and only if any continuous map $f : A \to Y$ can be extended to X.

6.11. Prove that if any continuous map $f : X \to X$ has a fixed point and A is a retract of X, then any continuous map $g : A \to A$ has a fixed point.

6.12. Let $\mathbb{S}^\infty$ be the set of all points $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$, such that only a finite number of x_i are nonzero and $\sum x_i^2 = 1$, supplied with the natural topology. Prove that the space $\mathbb{S}^\infty$ is *contractible* (i.e., homotopy equivalent to a point). *Hint:* Prove that the identity map is homotopic to the map $(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$.)