RESEARCH SUMMARY

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My research covers several topics within the theory of reductive algebraic groups. The notion of a simple, or, more generally, reductive algebraic group emerged in 1950s as a natural joint extension of the concepts of a simple Lie group and a finite simple group in the setting of algebraic geometry. Examples of such groups include classical matrix groups GL_n, SL_n, Sp_{2n} etc.; orthogonal groups $\mathrm{Spin}(q)$, $\mathrm{SO}(q)$, where q is a quadratic form; finite simple groups of Lie type; affine Kac-Moody groups.

Specifically, my object of study is an isotropic reductive group G over a general commutative ring R. The notion of isotropy originates from the theory of quadratic forms; in particular, the isotropic rank of $\mathrm{SO}(q)$ is the Witt index of q. Its group of points $G(R)$ contains a “large” elementary subgroup $E(R)$, given by explicit generators called elementary root unipotents. For example, the elementary subgroup of $\mathrm{GL}_n(R)$ is generated by the elementary matrices $\mathbf{1} + ae_{ij}$, $i \neq j$, $a \in R$. The quotient $K_1^G(R)$ of $G(R)$ by $E(R)$ is called the non-stable K_1-functor, or the Whitehead group of G. Earlier, I have proved that if k is a perfect field and G is defined by equations over k, the functor K_1^G is $\mathbb{A}^1$-invariant on the category of smooth k-algebras R, i.e. its value on R is the same as on the ring of polynomials over R. As a corollary, one deduces that K_1^G is the 0-th homotopy group of G in the sense of $\mathbb{A}^1$-homotopy theory of Morel–Voevodsky. I plan to extend these results to non-perfect k and “non-constant” groups G defined over R.

I also proved the centrality of the congruence kernel for $E(R)$, which is part of the so-called congruence subgroup problem, asking whether every normal subgroup of finite index in $G(R)$ corresponds to a finite index ideal in R. The proof involves describing $E(R)$-normalized subgroups in $G(R)$ generated by an elementary root unipotent. This leads naturally to the classification of all $E(R)$-normalized subgroups in $G(R)$, which is the goal of our ongoing joint work with A. Stepanov. Another related problem that I plan to address is verification of Kazhdan's property (T) for $E(R)$; this would extend the results of M. Ershov, A. Jaikin, and M. Kassabov.

Apart from the generators of $E(R)$, one would like to know a complete set of relations among them. The Steinberg group $\mathrm{St}(R)$ is defined as the group with the same set of generators as $E(R)$ but the only relations imposed are the "obvious" ones, such as the Chevalley commutator formula. Then one aims to show that the natural surjection of $\mathrm{St}(R)$ onto $E(R)$ is a universal central extension, and to compute its kernel $K_2^G(R)$ called the non-stable K_2-functor associated to G. If the group G is of isotropic rank 1, there is no universally accepted definition of a Steinberg group. In our recent joint paper with L. Boelaert and T. De Medts, we establish a one-to-one correspondence between adjoint reductive groups of isotropic rank 1 and isotopy classes of structurable division algebras over fields. I plan to extend this correspondence to isotropic groups over local rings (and, possibly, over any commutative rings), to obtain a construction of Steinberg groups in these terms, and to prove the centrality of $K_2^G(R)$. Together with S. Sinchuk and A. Lavrenov, we also plan to show that K_2^G is $\mathbb{A}^1$-invariant and coincides with the 1st $\mathbb{A}^1$-homotopy group of G, if G is a split group of type C_l, D_l, E_l, extending a theorem of M. Tulenbaev for GL_n.