SUMMARY OF THE PROJECT "MOTIVIC HOMOTOPY THEORY AND APPLICATIONS"

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Motivic homotopy theory is the homotopy theory of smooth algebraic varieties with the affine line $\mathbb{A}^1$ playing the role of the interval $[0, 1]$. The foundations were introduced by Morel and Voevodsky and now the field actively develops. The theory provides a flexible setting adjusted to applications of methods from the classical homotopy theory to the problems of algebraic geometry. The current project is aimed at the study of the general properties of the motivic stable homotopy category $\mathcal{SH}(k)$ as well as at its applications to the theory of cohomological invariants of algebraic groups.

Until recently there was an essentially unique way to construct $\mathcal{SH}(k)$ and this way lacked an effective tool to compute morphisms (roughly speaking, one takes an appropriate Bousfield localization of the category of presheaves of spectra on the category of smooth varieties). Following the ideas of Voevodsky (unpublished) Garkusha and Panin developed the theory of framed presheaves that provided a new description of $\mathcal{SH}(k)$ adjusted to computations. The theory of framed presheaves is a certain enhancement of the Voevodsky's theory of presheaves with transfers. Presheaves with transfers have a wrong-way maps (transfers) along the finite surjective morphisms; for framed presheaves one has transfers along finite surjective morphisms equipped with an additional data: a trivialization of the relative stable normal bundle. One can show that every generalized cohomology theory representable in $\mathcal{SH}(k)$ (such as Quillen K-theory, hermitian K-theory, various versions of algebraic cobordism, motivic stable cohomotopy groups, etc.) admits natural framed transfers. Calmès and Fasel proposed an alternative way to enhance the theory of presheaves with transfers bringing quadratic forms (which serve as a kind of an orientation) into the picture, obtaining the so-called presheaves with Milnor–Witt transfers. In a joint work with Alexander Neshitov we are going to study the relations between the framed presheaves and the presheaves with Milnor–Witt transfers and between the respective derived categories of motives. In particular, we are going to show that every linear framed Nisnevich sheaf that is homotopy invariant and stable admits a canonical structure of a presheaf with Milnor–Witt transfers.

As an another application of the framed techniques we are going to study the coefficient ring of the motivic MSL-cobordism. We would like to compute its algebraic diagonal. For the MGL-cobordism an analogous computation with the answer being the Lazard ring is a difficult theorem due to Hoyois–Hopkins–Morel. Moreover, we are going to investigate the 2-torsion in the coefficient ring of $\mathrm{MSL}[\eta^{-1}]$ in the case when the base field $k = \mathbb{R}$. Here η is the motivic Hopf map (in the motivic realm this map is not nilpotent). There is a theory of SL-orientation in algebraic geometry and this theory becomes especially neat if one inverts η: for example, the failure of the multiplicativity for the total Pontryagin classes is η-torsion (similar to the 2-torsion failure in the classical topology). So one could possibly expect the absense of the 2-torsion in the coefficient ring of $\mathrm{MSL}[\eta^{-1}]$.

As an application of the motivic homotopy theory we are going to describe the motive of a twisted form of the affine homogeneous variety E6/F4 (this is a joint work with Nikita Semenov). Our approach involves some computations with motivic Morava K-theories and the usage of a Morava variant of Brown-Gersten-Quillen spectral sequence. This computation is a manifestation of the conjectural relation between the vanishing of the cohomological invariants of algebraic groups and the computability of the Morava K-theory of the corresponding varieties.