SUMMARY OF THE RESEARCH STATEMENT

SERGEY RYBAKOV

A tower of algebraic curves over a filed k is an infinite sequence

$$\dots C_n \to C_{n-1} \to \dots \to C_0$$

of curves and finite morphisms. We assume that the genus $g(C_n)$ is unbounded. If $k = \mathbb{F}_q$ is a finite field, the number of points $|C_n(k)|$ on the curve C_n is defined, and the limit

$$\beta(C_\bullet) = \lim_{n\to\infty} \frac{|C_n(k)|}{g(C_n)}$$

exists. Moreover, by the Drinfeld-Vlăduţ theorem [TsVN07, 3.2.1], $\beta(C_\bullet) \le \sqrt{q}-1$. The tower is called *optimal*, if $\beta(C_\bullet) = \sqrt{q}-1$. It is known that if q is a square, examples of optimal towers over $\mathbb{F}_q$ can be constructed as modular towers [TsVZ82], and [I81], but in other cases all known towers are far from being optimal.

In the paper [Ry] we introduce a new construction of towers of algebraic curves over finite fields. As an example, we construct *the Legendre tower* starting from the Legendre family of elliptic curves over $\mathbb{F}_{p^2}$, and prove that it is an optimal tower.

Our construction is related to modular towers as follows. Recall that points on the curve $X_0(\ell^n)$ correspond to isomorphism classes of elliptic curves E with a cyclic subgroup in $E(\bar{k})$ of order ℓ^n. In fact, this curve is defined over $\mathbb{F}_{p^2}$ (and even over $\mathbb{F}_p$), and the family $X_0(\ell^\bullet)$ is an optimal tower over $\mathbb{F}_{p^2}$. Let $\mathcal{E} \to \mathbb{P}^1$ be a family of elliptic curves such that its j–invariant is map of degree 1. We expect that for an appropriate family $\mathcal{E}$ our construction gives the modular tower $X_0(\ell^\bullet)$. In particular, the Legendre tower is a base change of $X_0(\ell^\bullet)$.

We hope that our construction will help to find optimal towers over $\mathbb{F}_p$. In our research we focus on families of $K3$ surfaces coming from Fano threefolds. We are going to find families over $\mathbb{F}_p$ satisfying the following two properties: first, there are many strongly supersingular fibers in degree 2 over $\mathbb{F}_{p^2}$; second, the monodromy group is big in a reasonable sense. We say that a group G is *big* if there is an infinite subgroup $H \subset G$, and the quotient set G/H is also infinite. Such families correspond to towers of algebraic curves over $\mathbb{F}_p$ with good asymptotic properties.

References

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