- Project Summary -

Logical Aspects of Reasoning about Probability Spaces

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Probability logic is a branch of modern logic concerned with developing and studying various formal languages for reasoning about probabilistic structures — i.e. those having probabilistic components. Any language of this kind is called a *probability logic* as well. For each such logic $\mathscr{L}$ we assume the following:

— a collection $\mathtt{K}$ of probabilistic structures, called *$\mathscr{L}$-structures*,

— a collection $\mathtt{F}$ of string-like objects, called *$\mathscr{L}$-formulas*,

— a relation $\Vdash$ on $\mathtt{K} \times \mathtt{F}$, called the *satisfiability relation* for $\mathscr{L}$.

Different sorts of mathematical logicians (and theoretical computer scientists) — such as Scott, Keisler, Halpern, Terwijn, and others — approach the matter in different ways.

The present project is devoted to languages for reasoning directly about probability spaces. More precisely, I would like to focus on modifications of one very natural probability logic $\mathscr{L}^\star$ described in [Speranski 2017], which has two sorts of quantifiers, viz. over events and over real numbers. The project aims at studying these logics from different points of view — paying special attention to model-theoretic and computability-theoretic aspects. In fact, although certain fragments of $\mathscr{L}^\star$ interpreted over the class of all discrete probability spaces have been investigated in [Speranski 2017], arbitrary probability spaces (and their finitely additive analogues) are much harder to deal with.

From the computability-theoretic side, we want to measure the complexity and expressiveness of the logics for reasoning about probability spaces. Here various theories can be obtained in two essentially different ways, 'syntactical' and 'semantical':

i. by passing from the collection of all $\mathscr{L}^\star$-formulas to a suitable subcollection of it;

ii. by passing from the class of all probability spaces (or more generally, their finitely additive analogues) to a suitable subclass of it.

The study of these may be viewed as a topic in elementary theories and their fragments (see e.g. [Ershov et al. 1965] and [Nies 1996] for a survey), but with emphasis on complexity bounds instead of decidability or hereditary undecibility. Clearly real-valued functions are involved, so theories for reasoning about objects of analysis should be of particular interest to us. The pioneering results in this field of research are those of [Tarski 1951], and since then various results have been obtained by others; the role of $\mathscr{L}^\star$ for probability spaces is somewhat analogous to that of the languages of [Solovay et al. 2012] for real vector spaces and normed spaces.

From the model-theoretic side, we want to develop a theory of 'elementary' invariants of probability spaces which will give us a deeper understanding of the mathematical properties of quantified probability logics. More precisely, it should to be a refinement of the definition suggested in [Speranski 2017] — which was inspired by the general theory of Boolean algebras (cf. [Koppelberg 1989]) and has its uses, but doesn't, in general, provide enough information about probability measures. In particular, a good model-theoretic characterisation of spaces should give us a powerful tool for relating $\mathscr{L}^\star$ and its modifications to different formalisms that arise in the foundations of mathematics, as well as for obtaining sharper complexity bounds. Further, notice that we may understand the model-theoretic properties of some probabilistic theories better if we expand the language with countable conjunctions and disjunctions.

Also the present project will contribute to a better understanding of the relationship between computability and continuity — which evidently plays an important role in constructive mathematics — in the context of probability spaces. For instance, suppose that probability spaces (assumed countably additive) are replaced by their finitely additive analogues. Then the original notion of atomless and its variations lead to several weaker notions. Thus it is interesting to examine how the complexity of the theory of atomless spaces depends on our choice between these notions.

References

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