Framed motives over $\operatorname{Spec}\mathbb{Z}$.

The framed correspondences ware suggested by V. Voevodsky in [16] as the computational tool for the stable motivic homotopy category $\mathbf{SH}(S)$ [20],[17], [21, 18], that allows to reconstruct $\mathbf{SH}(S)$ as the category of so-called framed motives with the canonical precise construction of stable motivically fibrant resolutions. The original case of the theory is the case of (perfect) fields.

The theory was constructed by G. Garkusha and I. Panin [6], [8] basing on results of [7] and [1], [11] obtained in coautorship with A. Ananievsky and A. Neshitov, and with partition of other authors for a special cases of base fields [5],[3], [4], and continued with developments [12], [9]. Note the universal ∞-category $\mathbf{SH}^{\mathrm{fr}}(S)$ in [5].

One of the main results of [6] (Th 4.1) is the computation of the stable motivcally fibrant resolution of the suspension T-spectrum of presheaves on Sm_k in terms of presheaves of framed correspondences, see [6, sect 2].

The aim of the project is to get a formula in terms of framed correspondences in the category $\mathbf{SH}(\operatorname{Spec} Z)$.

More generally consider the case of $\mathbf{SH}(S,R)$ with the base scheme S of a Krull dimension d and the coefficient ring R being equal to $\mathbb{Z}$ if the residue fields of S are perfect, or the ring sheaf $\mathbb{Z}_{\mathrm{char}}$ that is the localisation of $\mathbb{Z}$ such that $l \in \mathbb{Z}^{\times}_{\mathrm{char}}(U) \Leftrightarrow l \notin \mathcal{O}^{\times}(U)$.

Consider the models of $\mathbf{SH}(S)$ (or $\mathbf{SH}(S,R)$) given by T-spectra of pointed presheaves $\mathbf{Pre}^{T}(\mathrm{Sm}_S)$ and $(S^1,\mathbb{G}_m)$-bi-spectra of pointed presheaves $\mathbf{Pre}^{S^1,\mathbb{G}_m}(\mathrm{Sm}_S)$. Denote by $L_{\text{st-mot}}$ the stable motivic localisation functors on the categories of the (bi-)spectra of presheaves, denote by $L_{\mathbb{A}^1}$ and L_{nis} the $\mathbb{A}^1$ and Nisnevich localisation functors, and denote by $L_{\mathbb{G}_m}$ the stable $\mathbb{G}_m$-localisation functor on the category of $(S^1,\mathbb{G}_m)$-bi-spectra.

So we are going to compute $L_{\text{st-mot}}$ in terms of the functors: $(-)^{\mathrm{fr}}$, $L_{\mathbb{A}}$, $L_{\mathbb{G}_m}$ and some "filtration"of L_{nis}.

Consider the following filtration on the Nisnevich topology on Sm_S over a base scheme S of dimension d:

$$\mathrm{nis} \supset \xi = \langle \phi_d, \dots \phi_1 \rangle \supset \langle \phi_{d-1}, \dots \phi_1 \rangle \supset \dots \phi_1 \supset \mathrm{triv}$$

where the topologies ξ and ϕ_d are defined as follows, and angle parentheses means the topology generated by the set, and triv denotes the trivial topology.

Definition 1. Let ξ be topology on Sch_S with coverings being etale morphisms $V \to U$ such that for any $x \in S$ there is a lift $U \times_S x \to V$.

The topology ξ is generated by the Nisnevich squares for the pairs of the form $(U, U \times_S Z)$ where U is an S-scheme and a closed subscheme in U that is equal to $U \times_S Z$ for an irreducible closed subscheme Z.

Define the topology φ_i on Sch_S as the topology generated by the Nisnevich squares for the pairs of the form $(U, U \times_S Z)$ where U is an S-scheme and a closed subscheme in U that is equal to $U \times_S Z$ for a closed subscheme Z of a pure codimension i.

Then the topology ξ is generated by the topologies φ_i for all integer i.

Theorem 1. *The canonical morphisms of (bi-)spectra*

$$\begin{aligned} L_{\text{st-mot}}(\Sigma^{\infty}_{S^1,\mathbb{G}_m} X) &= L_{\mathrm{nis}} L_{\mathbb{G}_m} L_{\mathbb{A}} L_{\phi_d} L_{\mathbb{A}} L_{\phi_{d-1}} \dots L_{\mathbb{A}} L_{\phi_1} L_{\mathbb{A}} (\Sigma^{\infty}_{S^1 \wedge \mathbb{G}_m} Y^{\mathrm{fr}}), \\ L_{\text{st-mot}}(\Sigma^{\infty}_{T} X) &= \varinjlim_i L_{\mathrm{nis}} \Omega^i_{\mathbb{G}_m} L_{\mathbb{A}} L_{\phi_d} L_{\mathbb{A}} L_{\phi_{d-1}} \dots L_{\mathbb{A}} L_{\phi_1} L_{\mathbb{A}} (\Sigma^{\infty}_{T} (\mathbb{G}_m^{\wedge i} \wedge Y)^{\mathrm{fr}}), \end{aligned}$$

are the equivalences in positive degrees with respect to T in the second case and with respect to S^1 in the first case.

For the case of $S = \operatorname{Spec} Z$ we have the formula

$$L_{\text{st-mot}}(\Sigma^{\infty} Y) = L_{\mathrm{nis}} L_{\mathbb{G}_m} L_{\mathbb{A}} L_{\xi} L_{\mathbb{A}} (\Sigma^{\infty} Y^{\mathrm{fr}}).$$

Corollary 1. *(Vanishing theorem) Let the base scheme S and the ring of coefficients R be as in the theorem. The (presheaf) stable motivic homotopy groups $\pi^s_{i,j}(Y,R)(U) = [U,Y]_{\mathbf{SH}(S,R)}$ vanishes for all $i < -\dim U$, and $U, Y \in \mathrm{Sm}_S$.*

Corollary 2. *(Stable connectivity theorem in $\mathbf{SH}(S,R)$.) Let the base scheme S and the ring of coefficients R be as above. The Nisnevich sheaf of stable motivic homotopy groups $\underline{\pi}^s_{i,j}(Y,R)$ vanishes for all $Y \in \mathrm{Sm}_S$, $i < -\dim S$.*

Remark 1. The vanishing theorems and connectivity theorems in the categories $\mathbf{SH}(S)$ and $\mathbf{SH}_{S^1}(S)$ are proven for the case of the base scheme with infinite residue fields, in the sequence [18], [19], [15], [2]. The results for the category $\mathbf{SH}(S)$ over an arbitrary base S follows form the case of a bases with infinite residue fields because of the arguments as in [5],[3].

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