Introduction into (projective) algebraic geometry

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The base field k is algebraically closed, unless specified otherwise. Characteristic is any, but sometimes we will have to assume that it is large enough (usually 5 is enough), or zero.

Completed in the Spring of 2020

- Introduction
  - Affine and projective varieties - definition and a few examples. Projective spaces. Veronese embedding.
- Algebraic curves - 1.
  - Plane curves, smooth and singular.
  - Plane conics are rational.
  - Smooth plane cubics are not rational.

    Singular plane cubics.
  - The group law on a plane cubic (elliptic curve). Case of the field of complex numbers (statement only).
- Algebraic surfaces - 1.
- Blow-up of a point on a plane.
- Intersection index of properly intersecting curves.
- Main theorem: invariance of the intersection index under the linear equivalence.
- Difficult part: self-intersection of a curve via linear equivalence. Examples: exceptional curve of the blow-up; resolution of a cone singularity.
- Singularities of surfaces - 1.
- Platonic solids in R^3 give some finite subgroups in the group PSL(2,C)
- Binary groups: lifting of a finite subgroup to the group SL(2).
- Example: binary icosahedral group after Felix Klein.
- Defining the factor-variety A^2 / G.
- Chevalley-Sheppard-Todd theorem: statement only, and some examples.
- Invariant theory: computing singular surfaces A^2 / G;
- Resolution of the singularities of surfaces.

- Example: resolution of the singularities x^2 + y^2 + z^2, x^2 + y^2 + z^3.
- Intersection of the components of the exceptional divisor. Difficult part: self-intersection of the components via linear equivalence.
- Example: Resolution of the icosahedral singularity x^2 + y^3 + z^5.

Remaining part

- Genus of a curve, definition via Poincare polynomial.
- Arithmetic genus of a smooth curve via cohomology, H^1(O) [1] . Equivalence with the POincare series definition.
- Linear equivalence of divisors.
- Picard group and the notion of Picard variety for a curve.
- Example: projective line, P^1.
- Example: elliptic curve (plane cubic): detailed study of linear systems.
  - Picard group of a surface: elementary examples: projective plane P^2, quadric P^1 x P^1.
- Abel-Jacobi morphism from the symmetric power of a curve to the Picard variety.
- Differential forms on curves.
- Residues of differential on a curve: definition only, without proof of invariance.
- Serre duality: sketch of the proof with resudes [2].
- Riemann-Roch theorem: sheaf-theoretic proof [3] .
- Hurwitz formula: algebraic proof.
- Case of complex numbers: Hurwitz formula: topological proof.
- Canonical embedding of a curve. Example: curves of genus 3 are plane curves of degree 4.
- Riemann-Roch theorem in a geometric form: linear span of a set of points in the canonical embedding.
- Hyper-elliptic curves. Curves of genus 2 are hyper-elliptic.
- Algebraic surfaces - 2.

[1] If I can assume that the notion of sheaf cohomology is known. I only need H^1 of a sheaf. Cech definition is enough.

[2] same

[3] same

- Adjunction formula for curves on a surface. Genus of a plane curve, again.
- First Chern class of a line bundle on a surface with values in the Picard group (as zeroes and poles of a rational section).
- Neron-Severi group of a surface. Examples: P^2, quadric.
- Digression: resolution of singularities of surfaces – some examples.
- Cubic surfaces in P^3.
  - Linear system of plane cubic curves through 6 points on P^2, and 27 lines: sketch of a proof. Picard group of a cubic surface is isomorphic to E_6.
  - Cremona transformations of P^2: Example.
- Grassmannian varieties - 1.
  - Grassmannian variety Gr(2,4).
  - Projective embedding of Gr(2,4). Plucker quadric.
  - Intersection theory on Gr(2,4). 27 lines, again. Idea of rational equavlence of cycles.