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TOPOLOGY-2

Math in Moscow Lecture Notes
Fall 2009

MCCME Publishers, Moscow 2009

TABLE OF CONTENTS

FOREWORD

The present booklet is based on lectures given in the fall semester of 2009 to second year IUM students in Russian and to Math in Moscow students in English. I prepared and distributed handouts, typeset in LaTeX, at the end of each lecture (the handouts, written in English, were given to the American students and to the Russian ones as well). These handouts, with slight revisions, are gathered in this small book.

The Topology-2 course at the IUM (and in the framework of the Math in Moscow program) is traditionally an introductory course in algebraic topology, mainly about homology theory. The students taking it have already had Topology-1, which at the IUM is an elementary introduction to topology with emphasis on its geometric and algebraic aspects. Topology-1 includes a minimal amount of general topology (topological spaces and continuous maps, topological equivalence, compactness, connectedness, separability) and such geometric topics as plane curves, surfaces, vector fields, covering spaces, examples of 3-manifolds, homotopy, treated by means of the corresponding basic algebraic invariants (degree of circle maps, Whitney winding number, Euler characteristic, Poincaré index, fundamental group, Morse index). For more details about that course, the reader is referred to the book *Topology-1* in the same series written jointly by V.V.Prasolov and myself.

As to the Topology-2 course, the titles of the 13 lectures will quickly give an idea of its contents. As usual, I regard the ability of solving the problems (appearing at the end of each lecture and used in the exercise class as well as for homework assignments) as at least as important as mastering the theory. In preparing the lectures, my main sources of information (besides my own memories of the subject) were the wonderful books *Course in Homotopic Topology* by A.T.Fomenko and D.B.Fuchs, V.V.Prasolov's *Homology Theory*, and S.V.Matveev's *Lectures on Algebraic Topology*.

Teaching the Topology-2 course in the fall and winter of 2009 was a very satisfying experience: the IUM class of some twenty students was very receptive and attentive, they corrected errors in the lectures and the handouts, asked very pointed questions, and politely complained whenever the proofs were only sketched. I am especially grateful to Vladimir Eisenshtadt and Daniel Le for numerous useful remarks and to Alexey Deinega (who helped prepare the first version of the illustrations). I would also like to thank Victor Shuvalov for the final version of the illustrations and ??? , for rapidly and efficiently editing this booklet.

Lecture 1
HOMOLOGY FUNCTORS

In this introductory lecture, we learn what homology theories (the main protagonists of this course) are and how they work in topology. Here we take their properties for granted (without actually constructing any of these theories) and use them to give simple proofs of some deep topological facts, thus motivating the study of homology.

1.1. Categories and functors

By definition, a *category* $\mathcal{C}$ is a pair consisting of a class $\mathrm{Obj}(\mathcal{C})$ of *objects* and, for any ordered pair of objects (X, Y), a set $\hom(X, Y)$ of *morphisms* with *domain* X and *range* Y; if $f \in \hom(X, Y)$, we write $f : X \to Y$ or $X \xrightarrow{f} Y$; for every ordered triple of objects joined by two morphisms f, g, $X \xrightarrow{f} Y \xrightarrow{g} Z$, a morphism, called the *composite* of f and g and denoted by gf (or $g \circ f$) is given; the objects and morphisms satisfy the two following axioms:

Associativity. If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, then

$$h(gf) = (hg)f : X \to W.$$

Identity. For every object Y there is a morphism $\mathrm{id}_Y : Y \to Y$ such that if $f : X \to Y$, then $\mathrm{id}_Y f = f$, and if $h : Y \to Z$, then $h\,\mathrm{id}_Y = h$.

If the morphisms $f : X \to Y$ and $g : Y \to X$ satisfy $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$, we say that f and g are *inverse* to each other, f (and g) are *isomorphisms*, and X and Y are *isomorphic*.

In the most common examples of categories, the objects are sets with an additional structure on them while the morphisms are maps preserving this structure (in some sense). In this course, we will be using the following examples of categories:

- $\mathcal{T}$op: topological spaces and their continuous maps;
- $\mathcal{S}$im: simplicial spaces and their simplicial maps;
- $\mathcal{CW}$: cell spaces (also called CW-complexes) and cellular maps;
- $\mathcal{AG}$r: Abelian groups and their homomorphisms;
- $\mathcal{GAG}$: graded Abelian groups and grading-preserving homomorphisms;

- $\mathcal{M}$od: modules over commutative rings and their homomorphisms;
- $\mathcal{V}$ect: vector spaces and linear operators;
- $\mathcal{F}$ld: fields and their homomorphisms;
- $\mathcal{S}$et: sets and their maps.
- $\mathcal{D}$ir: sets supplied with a partial order $\leq$ such that $\hom(X, Y)$ consists of the pair (X, Y) if $X \leq Y$ and is the empty set otherwise.

A *covariant functor* Φ from the category $\mathcal{C}$ to the category $\mathcal{D}$ assigns to every object X from $\mathcal{C}$ an object $\Phi(X)$ from $\mathcal{D}$ and to each morphism $f : X \to Y$ from $\hom(\mathcal{C})$, a morphism $\Phi(f) : \Phi(X) \to \Phi(Y)$ from $\hom(\mathcal{D})$ so that identical morphisms and composition are preserved, i.e.,

(i) $\Phi(\mathrm{id}_X) = \mathrm{id}_{\Phi(X)}$, (ii) $\Phi(gf) = \Phi(g)\Phi(f)$.

The two properties (i) and (ii) are often referred to as *functoriality*. Note that quite often the symbol Φ used here to denote both the assignment of objects and the assignment of morphisms is replaced by two different symbols for the different types of assignments.

A *contravariant functor* Ψ from the category $\mathcal{C}$ to the category $\mathcal{D}$ is defined similarly, except that "the arrows are reversed", i.e., $\Psi(f) : \Psi(Y) \to \Psi(X)$ and $\Psi(gf) = \Psi(f)\Psi(g)$.

Here are some examples of functors.

- The *forgetful functor* from the category $\mathcal{T}$op to the category $\mathcal{S}$et which assigns to every topological space its set of points (thus "forgetting" its topological structure) and to each continuous map the map itself ("forgetting" about its continuity).

- The contravariant functor from the category of compact Hausdorff spaces and their continuous maps to the category of Banach spaces and their continuous homomorphisms that assigns to each space X the (Banach) space of continuous real-valued functions on X.

- The covariant functor H_0 from $\mathcal{T}$op to $\mathcal{AG}$r such that $H_0(X)$ is the free Abelian group generated by the set of connected components of X, and if $f : X \to Y$ is a continuous map, then $H_0(f) : H_0(X) \to H_0(Y)$ is the homomorphism such that if C is a connected component of X and C' is the component of Y containing $f(C)$, then $H_0(f)(C) = C'$.

- A *direct system* on a category $\mathcal{C}$ is a covariant functor from the category of directed sets $\mathcal{D}$ir to the category $\mathcal{C}$. (This functor allows to define direct limits in an arbitrary category.)

The main protagonists of this course are homology theories, which are covariant functors from certain topological categories to certain algebraic ones, and we discuss one of them in a section of its own.

1.2. Homology theory as a functor

Homology (more precisely, *singular homology*) is a covariant functor from the category $\mathcal{T}op$ of topological spaces and their continuous maps to the category $\mathcal{GAG}$ of graded Abelian groups and grading-preserving homomorphisms. This means that to each topological space X and each positive integer $n \in \mathbb{N}$ an Abelian group (denoted $H_n(X)$) is assigned and for each continuous map $f : X \to Y$, a homomorphism $f_* : H_n(X) \to H_n(Y)$ is given.

The singular homology functor will be constructed (and its main properties established) in Lecture 7. At this stage, the only properties that we need, besides functoriality, i.e.,

$$(\mathrm{id}_X) = \mathrm{id}_{H_n(X)} \quad \text{and} \quad (f \circ g)_* = f_* \circ g_*$$

are the following isomorphisms

$$H_n(\mathbb{S}^n) \cong \mathbb{Z}, \quad H_n(\mathbb{D}^n) \cong 0 \quad \text{for any} \quad n \geq 1,$$

where $\mathbb{S}^n$ and $\mathbb{D}^n$ are the n-dimensional sphere and disk, respectively.

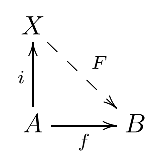
1.3. Diagram chasing and some basic problems of topology

One of the main notions of category theory (quite useful in other branches of mathematics as well) is the notion of commutative diagram. For example, a *commutative square* is a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow[f]{} & B \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow \\ C & \xrightarrow[g]{} & D \end{array}$$

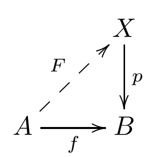
such that $\beta \circ f = g \circ \alpha$. More generally, a *commutative diagram* is a configuration of objects (e.g. groups) and arrows (morphisms) such that any two paths along arrows starting at the same place and ending at the same place have the same composition. Establishing the commutativity of a diagram and using it to derive certain properties of the morphisms involved is a mathematical sport called *diagram chasing* that we shall be practicing a bit in the exercise classes of this course.

The simplest commutative diagram is the commutative triangle, and it helps visualize some of the basic problems of topology, namely the extension problem (and its particular case, the retraction problem) and the lifting problem (and its particular case, the section problem). The *extension problem* is the following: given the (continuous) map $f : A \to B$ of a subset $A \subset X$ of a space X, to extend f to the whole space X, i.e., to find a map $F : X \to B$ such that $F(x) = f(x)$ for any $x \in A$ if such a map exists. The corresponding triangular diagram can be written in the form



where i denotes the inclusion $i(x) = x$ for all $x \in A$, while the dashed diagonal arrow is the map that we are searching for. The *retraction problem* is the particular case of the extension problem in which $A = B$ and $f =$id.

The *lifting problem* is as follows: given maps $f : A \to B$ and $p : X \to B$, to find the *lift* of f, i.e., a map $F : A \to X$ such that $p \circ F = f$. The corresponding triangular diagram has the form



where again the dashed diagonal arrow indicates the desired map. The *section problem* is the particular case of the lifting problem in which we have $f = \mathrm{id}_B : A = B \to B$ and $p : X \to B$ is a fiber bundle (a notion that will be discussed in the next lecture).

Homology theory often provides easy negative solutions to the basic problems by reducing them to very simple algebraic questions, in particular, for cases in which the direct topological (geometric) solutions are hopelessly difficult. We conclude this lecture by such an example. Many more will be discussed in the exercise class.

1.4. The retraction problem and Brouwer's fixed point theorem

Let us consider the retraction problem for the case in which $X = \mathbb{D}^{n+1}$, $A = \mathbb{S}^n = \partial\mathbb{D}^{n+1}$, f is the identity map, i is the inclusion $i : \partial\mathbb{D}^{n+1} \hookrightarrow \mathbb{D}^{n+1}$. Thus we are to find a retraction r of the $(n+1)$-disk to its boundary n-sphere.

Lemma. *There does not exist any retraction of the $(n+1)$-disk to its boundary n-sphere for any $n \in \mathbb{N}$.*

Proof. Suppose that such a retraction $r : \mathbb{D}^{n+1} \to \partial\mathbb{D}^{n+1}$ exists. Consider the maps $\mathbb{S}^n \overset{i}{\hookrightarrow} \mathbb{D}^{n+1} \xrightarrow{r} \mathbb{S}^n$. The corresponding n-homology homomorphisms are

$$H_n(\mathbb{S}^n) \overset{i}{\hookrightarrow} H_n(\mathbb{D}^{n+1}) \xrightarrow{r} H_n(\mathbb{S}^n).$$

By the definition of a retraction, $r \circ i = \mathrm{id}_{\mathbb{S}^n}$; by functoriality

$$(r \circ i)_* = r_* \circ i_* \quad \text{and} \quad (\mathrm{id}_{\mathbb{S}^n})_* = \mathrm{id}_{H_n(\mathbb{S}^n)};$$

but since $H_n(\mathbb{S}^n) = \mathbb{Z}$ and $H_n(\mathbb{D}^{n+1}) = 0$, we obtain the group homomorphisms

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{r_*} \mathbb{Z}$$

whose composition is the identity, which is impossible. □

An important application of this lemma is the famous Brouwer fixed point theorem.

Theorem (Brouwer). *Any continuous map f of the disk $\mathbb{D}^{n+1}$ has a fixed point, i.e., a point p such that $f(p) = p$.*

Proof. Suppose it doesn't. Then by setting $r(x)$ equal to the intersection of the ray $[f(x), x\rangle$ with the boundary sphere of the disk, we obviously obtain a retraction of the disk on its boundary sphere, which is impossible by the lemma. □

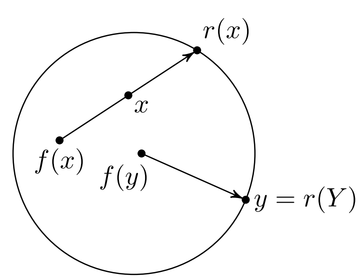


FIGURE 1. Proof of Brouwer's Theorem

1.5. Problems

1.1. Prove that any functor $T : \mathcal{C} \to \mathcal{D}$ assigns isomorphic objects to isomorphic objects.

1.2. In an arbitrary category define the notion of right inverse and left inverse of a morphism $f : X \to Y$ and show that if they both exist, then they coincide and f is an isomorphism. Prove that any isomorphism has a unique inverse.

In order to solve the subsequent problems, you can assume known the homology groups of $\mathbb{D}^n, \mathbb{S}^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{T}^2, M_g^2$.

1.3. Can the torus be retracted on its meridional circle?

1.4. Is there a retraction of $\mathbb{S}^5$ on $\mathbb{S}^4$, where $\mathbb{S}^4$ is the "equator" of $\mathbb{S}^5$, i.e. $\mathbb{S}^5 = \Sigma\mathbb{S}^4$ (here Σ denotes suspension) ?

1.5. Is there a retraction of $\mathbb{R}P^5$ on $\mathbb{R}P^4$, where $\mathbb{R}P^4 \hookrightarrow \mathbb{R}P^5$ is the natural embedding given by $(x^1; \ldots; x^5) \mapsto (x^1; \ldots; x^5; 0)$?

1.6. Is there a retraction of the solid torus $S = \mathbb{S}^1 \times \mathbb{D}^2$ on its boundary $dS = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$?

1.7. Given a continuous map $p : \mathbb{R}P^2 \to \mathbb{S}^1 \times \mathbb{D}^{17}$ and a homeomorphism $h : \mathbb{S}^1 \times \mathbb{D}^{17} \to \mathbb{S}^1 \times \mathbb{D}^{17}$, can h be lifted to a map $H : \mathbb{S}^1 \times \mathbb{D}^{17} \to \mathbb{R}P^2$?

1.8. Prove that Euclidean spaces of different dimensions are not homeomorhic.

Lecture 2

CW-COMPLEXES

Roughly speaking, a CW-complex is a topological space consisting of disks of various dimensions glued together so that the boundary of each disk is attached to disks of lower dimensions. In a certain sense, the category of of CW-complexes and their morphisms (called cellular maps) is the widest topological category that excludes unwieldy pathological topological spaces and has a reasonable geometric visualization. In this lecture, we prove the so-called cellular approximation theorem (which allows to replace arbitrary continuous maps by cellular ones) and begin the study of fiber bundles, one of the most important notions of geometric topology.

2.1. CW-complexes and their morphisms

Let X be topological space, X' its subset, and let $\sigma : \mathbb{D}^k \to X$ be a continuous map which is a homeomorphism on the interior $\operatorname{int}\mathbb{D}^k = \mathbb{D}^k \setminus \partial\mathbb{D}^k$ of the disk; then we say that σ is the *characteristic map* of a k-*cell* of X attached to the subset X'; we denote $e^k := \sigma(\text{int }\mathbb{D}^k)$, $\bar{e}^k := \sigma(\text{int }\mathbb{D}^k)$ and call these sets *open* and *closed* k-*cells*, respectively; the restriction $\sigma|_{\partial\mathbb{D}^k}$ is often called the *attaching map* of the k-cell $\bar{e}^k$. A CW-*space* (also called *cell space* or CW-*complex*) is defined as a topological space X presented as the union $X = \bigcup_k X^{(k)}$, where $X^{(k)}$ (called the k-*skeleton* of X) is the union of $X^{(k-1)}$ and a certain number of k-cells attached to $X^{(k-1)}$ provided that the two following axioms are satisfied:

- C (*closure finiteness*): the boundary $\bar{e}^k \setminus e^k$ of each k-cell intersects a finite number of l-cells of lower dimensions, i.e., cells with $l < k$;

- W (*weak topology*): a set $F \subset X$ is closed iff the intersection $F \cap \bar{e}^k$ is closed for any cell e^k.

If the number of cells constituting the CW-space is finite, the two axioms hold automatically. In that case (which we will usually consider) the CW-space X is obviously compact, finite-dimensional, and Hausdorff; it is useful to visualize it as the result of the following process: first take a finite number of points (the zero cells), attach a finite (possibly void) set of 1-cells (segments) by their endpoints to the set $X^{(0)}$ of zero cells, thus forming the 1-skeleton $X^{(1)}$, attach a finite (possibly void) set of 2-cells (2-disks) along their boundary circles to $X^{(1)}$, and so on, and finally attach the (finite nonempty) set of n-cells of the highest dimension n to $X^{(n-1)}$.

Here are some simple examples:

• The sphere $\mathbb{S}^n$, $n \geq 1$, has a CW-space structure with two cells.

• The disk $\mathbb{D}^n$, $n \geq 1$, has a CW-space structure with three cells.

• The torus $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ has a CW-space structure with four cells.

• The sphere with $h \geq 1$ handles M_g, $g = h-1$, has a CW-space structure with $4h+2$ cells.

More interesting examples will be studied in the exercise class.

A continuous map $f : X \to Y$ of CW-spaces is said to be a *cellular map* if $f(X^{(k)}) \subset Y^{(k)}$ for all $k \in \mathbb{N}$. The class of all CW-spaces supplied with cellular maps as morphisms forms one of the most important categories of modern topology, called the *category of CW-spaces.*

2.2. Cellular approximation

The cellular approximation theorem is a very useful tool in homology theory. It claims that any continuous map $f : X \to Y$ of CW-spaces is homotopic to a cellular map. We shall prove a more general statement, the so-called relative version of this theorem.

Theorem. *Let X and Y be CW-spaces, let $A \subset X$ be a CW-subspace (possibly $A = \varnothing$), and suppose that there exists a continuous map $f : X \to Y$ whose restriction to A is cellular. Then there exists a cellular map $g : X \to Y$ homotopic to f, and the homotopy on A is the identity.*

Proof. We proceed by induction on the dimension of the cells σ^n_α, extending the definition of the map g from the boundary of each cell to its interior. To do this, we look all the cells e^m_β, $m > n$, which contain part of the image of σ^n_α, and "blow out" this image to the boundary of e^m_β.

Remark. The image of a cell $\bar{e}_k$ in X can contain a cell of higher dimension $\bar{e}_m$, as the famous Peano curve example shows.

Thus, in order to prove the theorem, it suffices, given a continuous map $f : \mathbb{D}^n \to Y$ and the characteristic map of an m-cell $\chi : \mathbb{D}^m \to Y$, where $m > n$ and $f(\partial\mathbb{D}^n) \subset Y \setminus \mathrm{int}\chi(\mathbb{D}^m)$, to construct a map $g : \mathbb{D}^n \to Y$ such that

(i) if $f(x) \notin \mathrm{int}\chi(\mathbb{D}^m)$, then $g(x) = f(x)$;

(ii) the map g is homotopic to f, the homotopy being the identity outside $\mathrm{int}\chi(\mathbb{D}^m)$;

(iii) $g(\mathbb{D}^n) \subset Y \setminus \mathrm{int}\chi(\mathbb{D}^m)$.

Actually it suffices to prove that there exists a map $g : \mathbb{D}^n \to Y$ that satisfies conditions (i) and (ii) and whose image does not contain at least one point $y \in \mathrm{int}\chi(\mathbb{D}^m)$. Indeed, if this is the case, it is easy to use y to blow out the image of g to the boundary (see Fig.1); to do this, we consider the composition g_1 of the map $g = g_0$ constructed above with the "blow out" from y of the interior of $\mathbb{D}^m$ onto its boundary. This composition is homotopic to g_0, as can be seen from the formula

$$g_t(x) = (1-t)g_0(x) + tg_1(x)).$$

This will complete the proof, because g_1 now satisfies condition (iii) (as well as the conditions (i) and (ii)).

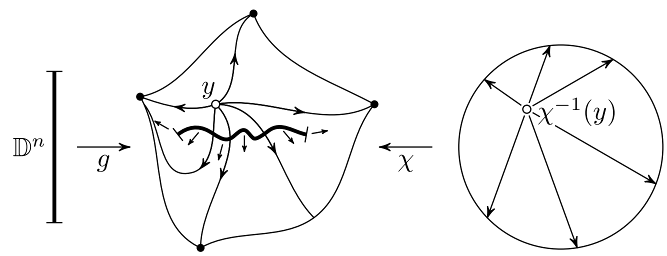


FIGURE 2. Blowing out the image of g

To construct g, we consider the system of concentric disks

$$\mathbb{D}^m_\varepsilon := \{x \in \mathbb{R}^m : \|x\| \le \varepsilon\},$$

so that $\mathbb{D}^m = \mathbb{D}^m_1$. For any $0 < \varepsilon < 1$, the disk $\mathbb{D}^m_\varepsilon$ is homeomorphic to $\chi(\mathbb{D}^m_\varepsilon) \subset Y$, and so we identify $\mathbb{D}^m_\varepsilon$ and $\chi(\mathbb{D}^m_\varepsilon) \subset Y$. The map f is uniformly continuous on the compact set $f^{-1}(\mathbb{D}^m_{3/4})$, so we can choose a $\delta > 0$ such that

$$x, y \in f^{-1}(\mathbb{D}^m_{3/4}) \subset \mathbb{D}^n \quad \& \quad \|x-y\| < \delta \Longrightarrow \|f(x) - f(y)\| < 1/4.$$

Take a triangulation of $\mathbb{D}^m$ all of whose simplices are of diameter less than δ. If the image under f of a simplex from this triangulation intersects the sphere $\mathbb{S}^{m-1}_{1/2} = \partial\mathbb{D}^m_{1/2}$, then this image is entirely contained in $\mathbb{D}^m_{3/4} \setminus \mathbb{D}^m_{1/4}$. The simplices of the triangulation split into three distinct classes:

(a) those whose images are disjoint from $\mathbb{S}^{m-1}_{1/2}$;

(b) those whose images are entirely contained in $\mathbb{S}^{m-1}_{1/2}$;

(c) those whose images intersect $\mathbb{S}^{m-1}_{1/2}$, but are not contained in it.

We shall construct the map g and the homotopy separately for each simplex. In case (a), we set $g(v) = f(v)$ for all vertices of the simplex and extend the map by linearity. In case (b), we don't change anything.

For a simplex whose image intersects $\mathbb{S}^{m-1}_{1/2}$ (case (c)), the situation is more complicated, because g is already defined on some of the faces (namely, on those satisfying (a) or (b)), and we must extend g to the entire simplex. For the vertices, we set $g(v) = f(v)$. On a 1-face, if the map is not yet defined, we extend it from the endpoints by linearity. On a 2-face Δ^2, if the map is not yet defined, we define it as follows: we cover Δ^2 by segments $[m, x]$ joining its baricenter m to points x on its boundary (at which g is already defined), then set $g(m) = f(m)$ and extend g to $[m, x]$ by linearity. Further, we perform the same construction for 3-faces, and so on, thus defining g on the entire disk $\mathbb{D}^n$.

A look at Fig.2, which shows only a "sector" of the disk $\mathbb{D}^m$, should help visualize what is going on.

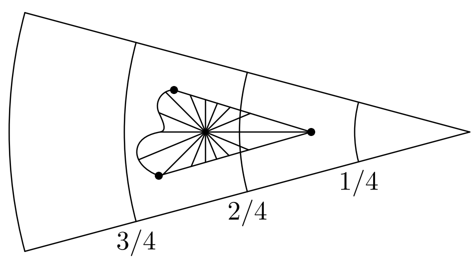


FIGURE 3. The image of Δ^k in $\mathbb{D}^m$

By construction, for any simplex Δ^k of the triangulation of $\mathbb{D}^n$, its image $g(\Delta^k)$ is contained in the convex hull of $f(\Delta^k)$. In case (c), this convex hull does not intersect $\mathbb{D}^m_{1/4}$. Indeed, if $y_0 \in (\Delta^k) \cap \mathbb{S}^{m-1}_{1/2}$, then $f(\Delta^k)$ lies in a disk of radius 1/4 centered at y_0, and this disk does not intersect $\mathbb{D}^m_{1/4}$. So the image $g(\Delta^k)$ has no points in $\mathbb{D}^m_{1/4}$.

The homotopy f_t between f and g is constructed as follows. Whenever $g(x) = f(x)$, we set $f_t(x) = f(x)$ for all t. If $g(x) \neq f(x)$, then both points $f(x)$ and $f(y)$ belong to $\mathbb{D}^m$ and we set

$$f_t(x) = (1-t)f(x) + tg(x)).$$

Now the intersection of $\mathbb{D}^m_{1/4}$ with the image of g lies in a finite number of affine planes of dimension $n < m$, so that the disk $\mathbb{D}^m_{1/4}$ must contain a point y nor belonging to this image, as required. □

2.3. Fiber bundles

A *locally trivial fiber bundle*, or a *fiber bundle* for short, is a quadruple (E, B, F, p), where E, B, and F are topological spaces, $p : E \to B$ is a surjective (continuous) map such that

• each point $x \in B$ has a neighborhood U homeomorphic to $U \times F$;

• the homeomorphism $U \times F \to p^{-1}(U)$ is compatible with p, i.e., the triangular diagram

$$\begin{array}{ccc} U \times F & \longrightarrow & p^{-1}(U) \\ & {\scriptstyle pr_1} \searrow & \downarrow {\scriptstyle p} \\ & & U \end{array}$$

in which pr_1 is the projection on the first factor, is commutative. The map p is called the *bundle projection*, B is the *base* of the bundle, F is the *fiber*, and E is the *total space* of the bundle.

Here are some examples of fiber bundles.

• The projection on the first factor of any Cartesian product is a fiber bundle, called *trivial*, the second factor playing the role of the fiber.

• Any covering space is a fiber bundle (with discrete fiber).

• The tangent bundle of an n-dimensional smooth manifold is a fiber bundle with fiber the n-dimensional linear space.

• The open (i.e., with boundary circle removed) Möbius band is the total space of a fiber bundle with base $\mathbb{S}^1$ and fiber $\mathbb{R}$.

• The natural map of the Stiefel manifold $V(k, n)$ onto the Grassmann manifold $G(k, n)$ is a fiber bundle with fiber the Lie group $\mathrm{GL}(k)$.

Note that all nontrivial fiber bundles have bases whose topology is, in some sense, nontrivial. This is no accident: any (locally trivial) fiber bundle is (globally) trivial if the base is "topologically trivial", e.g. is a disk, as the following statement shows.

Theorem [Feldbau]. *Any fiber bundle $p : E \to \mathbb{I}^k$ over the cube $\mathbb{I}^k$ is trivial, i.e., is the Cartesian product of the cube by the fiber F.*

Proof. First let us split the cube $\mathbb{I}^k = \mathbb{I}^{k-1} \times [0, 1]$ into two half-cubes $\mathbb{I}^k_1 = \mathbb{I}^{k-1} \times [0, 1/2]$ and $\mathbb{I}^k_2 = \mathbb{I}^{k-1} \times [1/2, 1]$, and, assuming that the bundles over the half-cubes $p_1 : E_1 \to \mathbb{I}^k_1$ and $p_2 : E_2 \to \mathbb{I}^k_2$ are trivial, prove that so is the bundle over the whole cube $\mathbb{I}^k$.

A point in E_1 has coordinates (x, f), $x \in \mathbb{I}_1^k$, $f \in F$. Let us denote similar coordinates in A_2 by $[x, f]$. If x belongs to both half cubes, then each point $e \in p^{-1}(x) \subset E_1 \cap E_2$, has coordinates (x, f_1) in E_1 and $[x, f_2]$ in E_2 (f_1 is not necessarily equal to f_2). This defines a map $f_x : F \to F$, $f_1 \mapsto f_2$. Let us map the half cubes onto their intersection via the natural projection

$$\pi : (x_1, \ldots, x_k) \mapsto (x_1, \ldots, x_{k-1}, 1/2)$$

and define the map $\varphi : E_2 \to \mathbb{I}_2^k \times F$ by the formula

$$\varphi[x, f] := (x, f_{\pi(x)}(f)).$$

Then φ is clearly a homeomorphism, which, together with the identical homeomorphism $E_1 \to \mathbb{I}_1^k \times F$, constitutes a homeomorphism $E \to \mathbb{I}^k \times F$; this shows that the bundle over the whole cube is trivial, as claimed.

Now let us divide the cube into halves again; then we can assume that the bundle over one of the halves is nontrivial (otherwise the theorem follows by what was proved above). Choosing such a nontrivial half-bundle $p^{(1)}$, we divide its base into two equal cubes and iterate the argument, choosing a nontrivial quarter-bundle $p^{(2)}$, and so on. After a certain number of iterations, the base of the chosen bundle $p^{(N)}$ will be so small that it will lie inside an open set over which the given bundle is trivial (here we are using the local triviality condition in the definition of fiber bundles), which is impossible since $p^{(n)}$ was assumed nontrivial. This contradiction proves the theorem. □

Of course fiber bundles form a category if one defines *bundle morphisms* in the natural way. Namely, suppose $\xi = (E, B, F, p)$ and $\xi' = (E', B', F', p')$ are two fiber bundles; then $\mu = (f, F)$ is a morphism, $\mu = (f, F) \in \hom(\xi, \xi')$, if the following square diagram

$$\begin{array}{ccc} E & \xrightarrow[F]{} & E' \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p'}\big\downarrow \\ B & \xrightarrow[f]{} & B' \end{array}$$

is commutative. We omit the definitions of bundle isomorphism and that of the notion of sub-bundle, leaving them to the reader.

Given a fiber bundle $p : E \to B$ and a map $f : X \to B$, one can define a bundle $p_f : E_f \to X$ over X, called the *pullback* of p via f, by setting

$$E_f := \big\{(e, x) \in E \times X \, : \, p(e) = f(x)\big\} \quad \text{and} \quad p_f(e, x) = p(x).$$

2.4. Covering homotopy theorem

Roughly speaking, the covering homotopy theorem says that the homotopy of any map to the base of a fiber bundle can be lifted to its total space, provided that a lift of the given map is specified. We shall prove a more general statement, the so-called relative version of the covering homotopy theorem. In it (and throughout this lecture), $\mathbb{I}$ denotes the closed interval $[0,1]$ and t is the parameter running over $\mathbb{I}$. Given a homotopy $H : X\times\mathbb{I} \to Y$, the notation $H_t : X \to Y$ will always be used for the map defined by the formula $H_t(x) := Y(x,t)$.

Theorem. *Suppose $p : E \to B$ is a fiber bundle, X is a CW-space, $X' \subset X$ a CW-subspace, and the following maps are given:*

- *a homotopy $H_t : X \to B$ and a lift $\widetilde{H}_0 : X \to E$ of its initial map H_0;*
- *a homotopy $\widetilde{H}'_t : X' \to E$ that covers the restriction H'_t of H_t to X';*

Then there exists a homotopy $\widetilde{H}_t : X \to E$ which covers the homotopy H_t and coincides with $\widetilde{H}'_t$ on X'.

Proof. We will need the following lemma (which is also useful in other contexts).

Lemma (Borsuk). *Suppose X is a CW-space, X' is a CW-subspace of X, and $f : X \to Y$ is a map. Then any homotopy $F' : X' \times \mathbb{I} \to Y$ of the map $f' := f|X'$ can be extended to a homotopy of f.*

Proof. We extend the homotopy by induction on the dimension n of the cells of the CW-space. Let $n = 0$; if $x_0 \in X'^{(0)}$, then the map $\{x_0\} \times \mathbb{I} \to Y$ is already defined, while if $x_0 \notin X'^{(0)}$, we send $\{x_0\} \times \mathbb{I}$ to $f(x_0)$.

Now suppose the homotopy has been extended to the skeleton X^n, where $n \geq 0$. Then for each $(n+1)$-cell, we have a map defined on $\mathbb{S}^n \times \mathbb{I}$ and on $\mathbb{D}^{n+1} \times \{0\}$ that we must extend to $\mathbb{D}^{n+1} \times [0,1]$. To do this, embed the cylinder $\mathbb{D}^{n+1} \times I$ in $\mathbb{R}^{n+2}$, choose a point P on the axis of the cylinder above its top $\mathbb{D}^{n+1} \times \{1\}$, and consider the projection $x \mapsto \varphi(x)$ from P of the cylinder to the union of its lateral surface $\mathbb{S}^n \times \mathbb{I}$ and its bottom $\mathbb{D}^{n+1} \times \{0\}$. Now send the point x to the image of $\varphi(x)$ under the map already defined by the inductive assumption. This proves the lemma by induction. □

We now return to the proof of the theorem. Three cases will be considered.

Case 1. Suppose that the bundle is trivial, i.e., $E = B\times F$ and $p(b,f) = b$. Then any map to E is the product of two maps (to B and to F), in particular we can write

$$\widetilde{H}'_t(x') = \big(H'_t(x'), \Phi_t(x')\big),$$

where $\Phi_t(x') \in F$ is the F-coordinate of the point $\widetilde{H}'_t(x')$. But Φ_t can be regarded as a homotopy to F given on a subset $X' \subset X$, and so, by Borsuk's Lemma, it can be extended to a homotopy $\widehat{\Phi}_t$ of the entire set X. Now if we put

$$\widetilde{H}_t(x) = \big(H_t(x), \widehat{\Phi}_t(x')\big)$$

we obtain the desired covering homotopy $\widetilde{H}_t$

Case 2. We now consider the case in which $p : E \to B$ is an arbitrary fiber bundle, but

$$X = \mathbb{D}^n \quad \text{and } X' = \partial\mathbb{D}^n = \mathbb{S}^{n-1}.$$

By assumption, we are given a homotopy $H_t : \mathbb{D}^n \to B$, a lift $\widetilde{H}_0 : \mathbb{D}^n \to E$, and a homotopy $\widetilde{H}'_t : \mathbb{S}^{n-1} \to E$ that covers the restriction H'_t of H_t to $\mathbb{S}^{n-1}$.

Let us take the pullback of the bundle p via the map H, obtaining the fiber bundle

$$p_1 : E_1 \to \mathbb{D}^n \times I = \mathbb{D}^{n+1},$$

where

$$E_1 = \big\{(d, e) \in \mathbb{D}^{n+1} \times E \;:\; p(e) = H(d)\big\}$$

and $p_1(d, e) = p(e)$. By the Feldbau Theorem, the bundle $H^*(p)$ is trivial.

Now consider the identity $\mathbb{D}^{n+1} \to \mathbb{D}^{n+1}$ as a homotopy $J_t : \mathbb{D}^n \to \mathbb{D}^{n+1}$, define

$$\widetilde{J}_0 \;:\; \mathbb{D}^n \to E_1$$

by setting

$$\widetilde{J}_0(d) = ((d, 0), H_0(d)),$$

denote $J'_t = J_t|_{\mathbb{S}^{n-1}}$, and define $\widetilde{J}'_t; \mathbb{S}^{n-1} \to E_1$ by setting

$$\widetilde{J}'_t(s) = ((s, 0), \widetilde{H}'_t(s)).$$

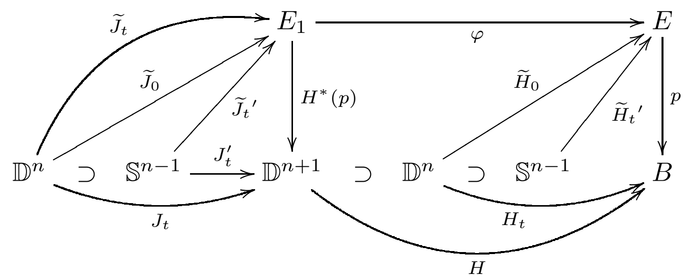
Then we are clearly in the situation of Case 1 with

$$X \rightsquigarrow \mathbb{D}^n, X' \rightsquigarrow \mathbb{S}^{n-1}, H_t \rightsquigarrow J_t, H'_t \rightsquigarrow J'_t, \widetilde{H}_0 \rightsquigarrow \widetilde{J}_0, \widetilde{H}'_t \rightsquigarrow \widetilde{J}'_t.$$

Using the result of Case 1, we obtain a homotopy $\widetilde{J}_t : \mathbb{D}^n \to E_1$.

Now we conclude Case 2 by defining $\widetilde{H}_t : \mathbb{D}^n \to E$ via the formula $\widetilde{H}_t(d) := \varphi(\widetilde{J}_t(d))$, where $\varphi : E_1 \to E$ is the map $(d, e) \mapsto e$. The verification of the required properties of H_t is straightforward from the construction.

The above construction is summarized by the following diagram:



Case 3. We now suppose that the bundle $p : E \to B$ and the pair (X, X') are arbitrary. The proof readily proceeds by induction on the dimension n of the cells constituting $X^n \subset X$, using the results of Case 2 to define the required map for each n-cell. □

2.6. Problems

2.1. Let C be the union of all circles of center $(1/n, 0)$ in the xy-plane and radius $1/n$. Prove that C is not homeomorphic to a CW-space.

2.2. a) Find a space satisfying (W) but not (C).
b) Find a space satisfying (C) but not (W).
c) Is the closure of cell necessarily a subspace ?

2.3. Find mimimal CW-complex structure on $\mathbb{C}P^n$, $\mathbb{R}P^n$, $\#_{i=1}^q \mathbb{T}_i^2$, $\#_{i=1}^q \mathbb{R}P_i^2$.

2.4. Define $\mathbb{S}^\infty$, $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$ and supply them with CW-structure.

2.5. Prove that $\mathbb{S}^\infty$ is contractible.

2.6. Give an example of a nontriangulable two-dimensional CW-complex.

2.7. Prove that $\pi_1 X$ lives in $X^{(2)}$ (the 2-skeleton of X), i.e., show that $\pi_1(X) \equiv \pi_1(X^{(2)})$.

2.8. Prove that a CW-complex is connected iff its 1-skeleton $X^{(1)}$ is connected.

2.9. Prove that a CW-complex is connected iff it is path-connected.

2.10. Prove that $\pi_k \mathbb{S}^n = 0$ for all $k < n$.

2.11. Show that the Cartesian product, the cone, the suspension, and the join of CW-complexes have natural CW-complexes structures.

2.12. Prove that any finite CW-complex X^n can be embedded in $\mathbb{R}^N$, where $N = (n+1)(n+2)/2$.

2.13. Show that $\pi_n(\mathbb{S}^1)$ are trivial for all $n > 1$. (Hint : Consider the bundle $p : \mathbb{R}^1 \to \mathbb{S}^1$ whose projection is given by formula $p(x) = \exp(xi)$)

2.14. Prove that a (locally trivial) fiber bundle $p : \mathbb{S}^n \to B$ whose base B consists of more than one point is not homotopic to a constant map.

Lecture 3

HOMOTOPY GROUPS

Like homology theory, homotopy group theory is a functor from the category of topological spaces and their (continuous) maps to the category of graded groups and their homomorphisms. The construction of homotopy groups is much simpler than that of homology groups and they have many of the properties of homology groups, so that they could be used to solve topological problems in the spirit of Lecture 1. Unfortunately, they are much more difficult to compute than the homology groups, and this restricts their applications. Thus the computation of the homotopy groups of spheres is still an open problem (despite over 50 years of efforts by the world's best topologists), but we shall compute one of them $\pi_3(\mathbb{S}^2)$, which will give us the occasion to discover (rediscover?) the beautiful Hopf bundle.

3.1. Homotopy group theory as a functor

We shall now define the homotopy groups $\pi_n(X, x_0)$ of an arbitrary topological space X with fixed base point $x_0 \in X$ for all $n \geq 2$. In the case $n = 1$, the group $\pi_1(X, x_0)$ is the fundamental group, which we assume known.

Let (X, x_0) be a topological space supplied with a base point, and let $(\mathbb{I}^n, \partial\mathbb{I}^n)$ the pair consisting of the n-cube and its (topological) boundary sphere. The group $\pi_n(X, x_0)$, $n \geq 2$, is defined as the set of *spheroids*, i.e., base point preserving homotopy classes of maps $\alpha : (\mathbb{I}^n, \partial\mathbb{I}^n) \to (X, x_0)$, i.e. continuous maps such that $\alpha(\partial\mathbb{I}^n) = x_0$ and homotopies H_t such that $H_t(\partial\mathbb{I}^n) = x_0$ for all $t \in \mathbb{I}$. (The explanation for the term "spheroid" is that $(\mathbb{I}^n/\partial\mathbb{I}^n) = \mathbb{S}^n$, see Fig.3.1.)

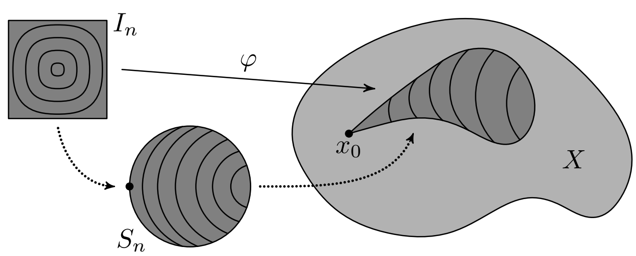


FIGURE 3.1. Spheroids

Further we define the *product* of two spheroids as shown in Fig.2, in which the shaded part of the cube $\mathbb{I}^n$ is mapped to the base point of $x_0 \in X$.

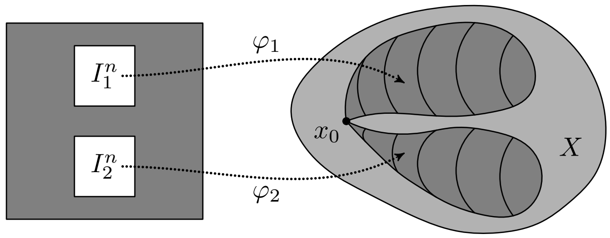


FIGURE 3.2. Product of two spheroids

Proposition. *The set* $\pi_n(X, x_0)$, $n \geq 2$, *with the product operation as defined above, is an Abelian group.*

Proof. It is easy to see that the neutral element in $\pi_n(X, x_0)$ is the constant spheroid $\mathbb{S}^n \to x_0$. For the inverse element to an arbitrary spheroid $\alpha : (\mathbb{I}^n, \partial\mathbb{I}^n) \to (X, x_0)$, we take the map $\alpha'(x, s) := \alpha(x, -s)$; then the composition is homotopic to the constant map, as will be shown in the exercise class.

If $n \geq 2$, the product operation is commutative. To see this, it suffices to take a good look at Fig.3.

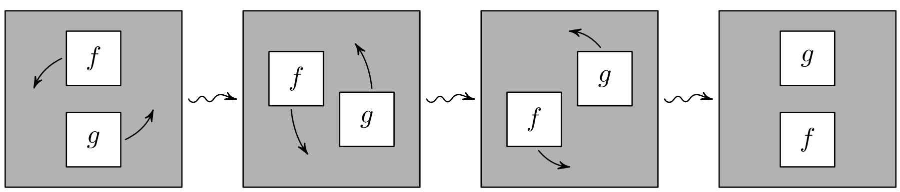


FIGURE 3.3. Commutativity of the product operation

Finally, associativity can readily be shown (it will probably be discussed in the exercise class). □

Of course homotopy groups constitute a functor, and we now define, for a given base point preserving map $f : (X, x_0) \to (Y, y_0)$ of topological spaces and any $n \geq 2$, the corresponding *induced homomorphism*

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0).$$

This is done in the natural way, i.e., by setting $f_* := \{f \circ \alpha\}$ for any spheroid $\{\alpha\} \in \pi_n(X, x_0)$, where the curly brackets denote base point preserving homotopy classes. The fact that f_* is well defined (i.e., does not depend on the choice $\alpha \in \{\alpha\}$) is a straightforward verification.

The proof of the following theorem is also a straightforward verification.

Theorem. *The homotopy groups are homotopy invariant, i.e.,*

$$X \simeq Y \Longrightarrow \pi_n(X, x_0) \cong \pi_n(Y, y_0) \quad \text{for all} \quad n > 0.$$

As pointed out above, although the homotopy groups are easy to define, they are hard to compute. Here are some examples and properties:

- $\pi_n(\text{point}) = 0$ for all $n \geq 0$;
- $\pi_k(\mathbb{S}^n) = 0$ for all $k < n$ (this easily follows from the Cellular Approximation Theorem);
- $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ for all $n \geq 1$ (this is a consequence of the Hurewicz Theorem, which will be proved in subsequent lectures, and the fact that $H_n(\mathbb{S}^n) = \mathbb{Z}$);
- $\pi_n(X \times Y) = \pi_n(X) \oplus \pi_n(Y)$ (obvious);

An important property of the homotopy groups is the fact that *there exists a natural action of the fundamental group* $\pi_1(X, x_0)$ *on* $\pi_n(X, x_0)$, $n \geq 2$. Because of this action, $\pi_2(\mathbb{S}^1 \wedge \mathbb{S}^2)$ is not isomorphic to $\mathbb{Z}$, as one might naively suppose. Indeed, we can modify the spheroid id $:\mathbb{S}^2 \to \mathbb{S}^2$ by first mapping $\mathbb{S}^2$ to the wedge $\mathbb{S}^2 \wedge \mathbb{I}$ by identifying the parallels within the Arctic Circle to points and then wrapping the "tail" $\mathbb{I}$ around $\mathbb{S}^1$ several times. The element of $\pi_2(\mathbb{S}^1 \wedge \mathbb{S}^2)$ thus obtained differs from the spheroid id. In fact $\pi_2(\mathbb{S}^1 \wedge \mathbb{S}^2) \cong \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \ldots$.

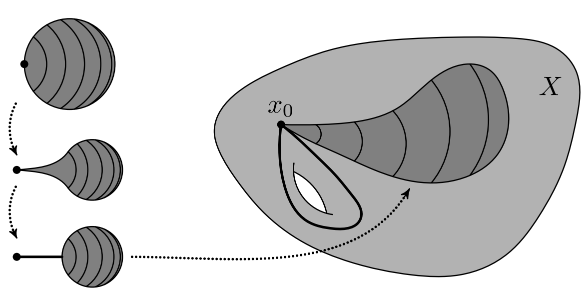


FIGURE 3.4. The action of $\pi_1(X)$ on $\pi_n(X)$

3.2. Exact homotopy sequence for fiber bundles

A sequence of groups and homomorphisms

$$\ldots \longrightarrow G_i \xrightarrow{\varphi_i} G_{i-1} \xrightarrow{\varphi_{i-1}} G_{i-2} \longrightarrow \ldots$$

is called *exact at the term* G_{i-1} if $\mathrm{Im}(\varphi_i) = \mathrm{Ker}(\varphi_{i-1})$, and simply *exact* if it is exact at all its terms. It is easy to prove (see the exercise class) that in any exact sequence of the form

$$0 \longrightarrow G_2 \xrightarrow{\varphi} G_1 \longrightarrow 0$$

the homomorphism φ is necessarily an isomorphism.

Now suppose $p : E \to B$ is a fiber bundle with fiber F, e_0 and b_0 are base points such that $p(e_0) = b_0$ and $p^{-1}(b_0) = F \ni e_0$. Denote by i the inclusion $(F, e_0) \hookrightarrow (E, e_0)$. We already have two of the homomorphisms that will appear in our sequence (namely p_* and i_*); we now construct the third one

$$\partial_* : \pi_n(B, b_0) \to \pi_{n-1}(f, e_0)$$

as follows. Let $\alpha : (\mathbb{S}^n, s_0) \to (B, b_0)$ be a spheroid from $\pi_n(B, b_0)$. Think of the sphere $\mathbb{S}^n$ as lying in $\mathbb{R}^{n+1}$ and cut it up into $(n-1)$-spheres with common point the North Pole s_0 by rotating $(n-1)$-dimensional hyperplanes passing through s_0. Then the map α can be regarded as a homotopy $\alpha_t : \mathbb{S}^{n-1} \to B$ joining two copies α_0, α_1 of the constant map $\mathbb{S}^{n-1} \to b_0 \in B$ (see Fig.5).

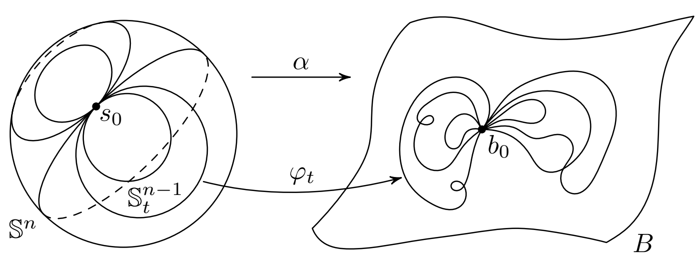


FIGURE 3.5. Definition of ∂_*

By the covering homotopy theorem, there exists a covering $\widetilde{\varphi}_t : \mathbb{S}^{n-1} \to B$. We define ∂_* by setting $\partial_*(\alpha) := \widetilde{\varphi_1}$.

Theorem. *The sequence of homomorphisms*

$$\cdots \to \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\partial_*} \pi_{n-1}(F, e_0) \to \dots$$

defined above is exact.

Proof. To show the exactness of our sequence, we must prove six inclusions of the form Im(.)⊂ Ker(.) and Ker(.)⊂ Im(.). They follow from the definitions and the Covering Homotopy Theorem; we shall prove only one inclusion (namely $\mathrm{Ker}(\partial_*) \subset \mathrm{Im}(p_*)$), leaving the other ones for the exercise class.

We can represent any spheroid $\alpha_0 : (\mathbb{S}^n, s_0) \to (B, b_0)$ as a homotopy $\alpha_t : \mathbb{S}^{n-1} \to B$ and (by the Covering Homotopy Theorem) and consider the covering homotopy $\widetilde{\alpha}_t : \mathbb{S}^{n-1} \to E$. If α_0 was chosen in $\mathrm{Ker}(\partial_*)$, then $\widetilde{\alpha}_1 : \mathbb{S}^{n-1} \to F$ is homotopic to the constant map. Let β_s be the homotopy in F joining $\widetilde{\alpha}_1$ to the constant map. Consider the homotopy

$$\widetilde{\psi}_t = \begin{cases} \widetilde{\alpha}_{2t} & \text{if } t \in [0, 1/2], \\ \widetilde{\beta}_{2t-1} & \text{if } t \in [1/2, 1]. \end{cases}$$

Corresponding to this spheroid is the spheroid $\widetilde{g} : \mathbb{S}^{n-1} \to F$ for which the map $g := p \circ \widetilde{g}$ is homotopic to α_0. Hence $\alpha_0 \in \mathrm{Im}\ p_*$, as required. □

3.3. Exact homotopy sequence for pairs

We shall now define the homotopy groups $\pi_n(X, A, a_0)$ of a pair of topological space with base point $a_0 \in A \subset X$ for all $n \geq 2$. Define a *relative spheroid* as a base point preserving homotopy class of maps

$$\alpha : (\mathbb{D}^n, \partial\mathbb{D}^n, s_0) \to (X, A, x_0),$$

i.e., continuous maps such that $\alpha(s_0) = x_0$ and homotopies H_t such that $H_t(\partial\mathbb{D}^n) \subset A$ and $H_t(s_0) = x_0$ for all $t \in \mathbb{I}$. It is often more convenient to interpret spheroids as (classes of) maps $(\mathbb{I}^n, \partial\mathbb{I}^n, s_0) \to (X, A, a_0)$ (this interpretation gives the same result, because the pair $\mathbb{I}^n/\partial I^n$ is homeomorphic to $(\mathbb{D}^n, \partial\mathbb{D}^n)$ The main definition, that of the *product* of two relative spheroids is relegated to the exercise class, together with the proof of the fact that $\pi_n(X, A, a_0)$ *is a group for any* $n \geq 2$ *and is Abelian for* $n \geq 3$.

Of course the homotopy group $\pi_n(\cdot, \cdot)$ of pairs can be regarded, for $n \geq 3$, as a functor from the category of pairs of topological spaces to the category of Abelian groups; the main definition, that of the *induced homomorphism*

$$f_* : \pi_n(X, A, a_0) \to \pi_n(Y, B, b_0)$$

corresponding to the map of pairs $f : (X, A) \to (Y, B)$, is left as an exercise.

Given a pair of spaces with base point, (X, A, a_0), we have the inclusion map $i : (A, a_0) \hookrightarrow (X, a_0)$, which determines the induced homomorphism i_*. Since any absolute spheroid $(\mathbb{D}^n, s_0) \to (X, a_0)$ can be regarded as a relative spheroid $(\mathbb{D}^n, \mathbb{S}^{n-1}, s_0) \to (X, A, a_0)$ (which takes $\mathbb{S}^{n-1}$ to a_0), we also have the homomorphism $p_* : \pi_n(X, a_0) \to \pi_n(X, A, a_0)$. Finally, we define

$$\partial_* : \pi_n(X, A, a_0) \to \pi_n(A, a_0)$$

by assigning to the relative spheroid $(\mathbb{D}^n, \mathbb{S}^{n-1}, s_0) \to (X, A, a_0)$ its restriction to $\mathbb{S}^{n-1}$.

Theorem. *The sequence of homomorphisms*

$$\dots \to \pi_n(A, a_0) \xrightarrow{i_*} \pi_n(X, a_0) \xrightarrow{p_*} \pi_n(X, A, a_0) \xrightarrow{\partial_*} \pi_{n-1}(A, a_0) \to \dots$$

defined above is exact.

Proof. Again we must verify six inclusions, but we will check only one, namely $\operatorname{Im} p_* \subset \operatorname{Ker} \partial_*$. Let $\alpha : \mathbb{I}^n \to X$ be a spheroid whose restriction to $\mathbb{I}^{n-1}$ is homotopic to the constant map to a_0 (in the class of maps $\mathbb{I}^{n-1} \to A$). Let $g_t : \mathbb{I}^{n-1} \to A$ be a homotopy between the restriction of α to $\mathbb{I}^{n-1}$ and the constant map. Consider the homotopy $f_t : \partial\mathbb{I}^n \to X$ that coincides with g_t on $\mathbb{I}^{n-1}$ and takes $\partial\mathbb{I}^n \setminus \mathbb{I}^{n-1}$ to a_0. Using the Borsuk Lemma, we extend this homotopy to a homotopy of α. Thus we obtain a homotopy in the class of relative spheroids that joins α to a relative spheroid taking $\partial\mathbb{I}^n$ to a_0. □

3.4. The Hopf bundle and $\pi_3(\mathbb{S}^2)$

The *Hopf bundle*, one of the most beautiful constructions in topology and in all of mathematics, is the fiber bundle $h : \mathbb{S}^3 \to \mathbb{S}^2$ constructed in the following way: present the sphere $\mathbb{S}^3$ in the form

$$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1|^2 + |z_2|^2 = 1\}$$

and define an action of the circle $\mathbb{S}^1 = \{w \in \mathbb{C} : |w| = 1\}$ by the rule $(z_1, z_2) \mapsto (wz_1, wz_2)$; it is easy to prove (see the exercise class) that the orbit space of $\mathbb{S}^3$ under this action is the 2-sphere.

This definition is very simple but hard to visualize. A more visual definition, based on the fact that the 3-sphere can be obtained by gluing together two solid tori, will be discussed in the exercise class, but the best way to visualize the Hopf bundle is to look at the marvelous animation of the construction due to Etienne Ghys (see "Dimensions" on his web page).

The Hopf bundle can be used to obtain the following remarkable formula for $\pi_3(\mathbb{S}^2)$, which in its day was a little mathematical sensation:

$$\boxed{\pi_3(\mathbb{S}^2) = \mathbb{Z}.}$$

To prove this, let us write out part of the homotopy sequence for the Hopf bundle:

$$\cdots \to \pi_2(\mathbb{S}^3) \xrightarrow{p_*} \pi_2(\mathbb{S}^2) \xrightarrow{\partial_*} \pi_1(\mathbb{S}^1) \xrightarrow{i_*} \pi_1(\mathbb{S}^3) \to \dots .$$

Since $\pi_2(\mathbb{S}^3) = \pi_1(\mathbb{S}^3) = 0$ (see the previous section), we obtain the isomorphism $\pi_2(\mathbb{S}^2) \cong \pi_1(\mathbb{S}^1)$, and since the fundamental group of the circle is $\mathbb{Z}$, we have $\pi_2(\mathbb{S}^2) \cong \mathbb{Z}$. (This is particular case of a fact mentioned but not proved in the previous section.)

Now let us write out another part of the same homotopy sequence for the Hopf bundle:

$$\cdots \to \pi_3(\mathbb{S}^1) \xrightarrow{i_*} \pi_3(\mathbb{S}^3) \xrightarrow{p_*} \pi_3(\mathbb{S}^2) \xrightarrow{\partial_*} \pi_2(\mathbb{S}^1) \to \dots$$

Since the two extreme terms are zero, we obtain $\pi_3(\mathbb{S}^3) \cong \pi_3(\mathbb{S}^2)$, and our claim follows from the isomorphism (mentioned in the previous section for any n, not only 3) $\pi_3(\mathbb{S}^3) \cong \mathbb{Z}$. □

3.5. Homotopy groups of spheres: some information

The computation of $\pi_k(\mathbb{S}^n)$ for $k > n$, arguably a useless task, was one of the central topics of mathematics in the 1950ies and 1960ies, mostly out of sheer curiosity. The remarkable work of L.S.Pontryagin, J.-P.Serre, and J.F.Adams eventually turned out to have other, much more useful applications. Although open questions still remain, the computation of the homotopy groups of spheres is no longer very fashionable today. Here we list only a few results, chosen more or less at random, that may seem very strange at first glance.

- $\pi_n(\mathbb{S}^n) = \mathbb{Z}, \quad n \geq 1;$
- $\pi_{n+1}(\mathbb{S}^n) = \mathbb{Z}_2, \quad n \geq 3;$
- $\pi_{n+2}(\mathbb{S}^n) = \mathbb{Z}_2, \quad n \geq 4;$
- $\pi_{n+3}(\mathbb{S}^n) = \mathbb{Z}_{24}, \quad n \geq 5;$
- $\pi_{n+4}(\mathbb{S}^n) = 0, \quad n \geq 6;$
- $\pi_{n+7}(\mathbb{S}^n) = \mathbb{Z}_{240}, \quad n \geq 9;$
- $\pi_{n+9}(\mathbb{S}^n) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad n \geq 11;$
- $\pi_{n+11}(\mathbb{S}^n) = \mathbb{Z}_{504}, \quad n \geq 13.$

The main tool for proving these results are the so-called spectral sequences, in particular those due to J.-P.Serre and J.F.Adams. Spectral sequences are a very sophisticated and effective means for computing homology groups, but are beyond the framework of this course.

3.6. Problems

3.1. Prove that $\pi_k X$ lives in $X^{(k+1)}$, more precisely $i_k : X^{(k)} \hookrightarrow X$ induces an isomorphism $(i_k)_* : \pi_k X^{(k+1)} \to \pi_k X$ for $k < n$ and an epimorphism for $k = n$ (n is the dimension of X)

3.2. Prove that $\pi_n(\mathbb{S}^1 \vee \mathbb{S}^n) = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \ldots$.

3.3. Prove that the product of spheroids is well defined.

3.4. Prove that the induced homomorphism $f_* : \pi_n X \to \pi_n Y$ is well defined.

3.5. Prove the exactness of the homotopy sequence for fiber bundles at the other terms.

3.6. Prove the exactness of the homotopy sequence for pairs at the other terms.

3.7. Prove that $\mathbb{C}P^2 = \mathbb{D}^4 \cup \mathbb{C}P^1$, where $p : \mathbb{S}^3 \to \mathbb{C}P^1$ is the Hopf bundle and $\mathbb{S}^3 = \partial \mathbb{D}^4$.

3.8. Does there exist a retraction $r : \mathbb{C}P^2 \to \mathbb{C}P^1$, where $\mathbb{C}P^1$ is embedded in $\mathbb{C}P^2$ in the natural way?

3.9. If A is a retract of X, then
a) the map $i_* : \pi_n(A) \to \pi_n(X)$ is injective ;
b) the map $p_* : \pi_n(X) \to \pi_n(X, A)$ is surjective;
c) the map $\partial_* : \pi_n(X, A) \to \pi_{n-1}(A)$ is a zero homomorphism;
d) $\pi_n(X) = \pi_n(X, A) \oplus \pi_n(A)$.

3.10. If there exists a homotopy $f_t : X \to X$ for which we have $f_0 = id$ and $f_1(X) \subset A$, then
a) the map $i_* : \pi_n(A) \to \pi_n(X)$ is surjective ;
b) the map $p_* : \pi_n(X) \to \pi_n(X, A)$ is a zero homomorphism;
c) the map $\partial_* : \pi_n(X, A) \to \pi_{n-1}(A)$ is injective;
d) $\pi_n(A) = \pi_{n+1}(X, A) \oplus \pi_n(X)$.

Lecture 4

CELLULAR HOMOLOGY

Cellular homology theory is a functor from the category of CW-complexes to graded Abelian groups that has the advantage (over other homology theories) of greatly simplifying the actual computations of the homology groups for the basic topological spaces, e.g. manifolds. In this lecture, we are rather high-handed about providing rigorous proofs of the properties of this functor, so as to be able to perform these computations as quickly as possible. The theory is based on the notion of degree of a map $f : \mathbb{S}^n \to \mathbb{S}^n$ of the sphere to itself, with which we begin.

4.1. Degree of self maps of the n-sphere

The simplest way to define the degree of a map $f : \mathbb{S}^n \to \mathbb{S}^n$ is to set $\deg(f) ;= f_*(1) \in \mathbb{Z}$, where $f_* : \pi_n(\mathbb{S}^n) \to \pi_n(\mathbb{S}^n) = \mathbb{Z}$ is the homomorphism induced by f. But we can't do that, because we have *not established* that $\pi_n(\mathbb{S}^n) = \mathbb{Z}$, a fact which is usually proved by using homology theory (specifically, the Hurewicz theorem, to appear in Lecture 6), a theory that we are only *beginning* to construct; thus such an approach would constitute a logical vicious circle. So we need a geometric definition of $\deg(f)$, and will give it in the case of a rather restricted class of maps f.

Let us call a continuous map $\varphi : \mathbb{S}^n_1 \to \mathbb{S}^n_2$ *neat* if, for some point $p \in \mathbb{S}$ the set $\mathbb{S}^n_1 \setminus \varphi^{-1}(p)$ consists of a family (possibly empty) or disjoint open balls $B^n_1, \dots, B^n_k$ each of which is mapped by φ diffeomorphically onto $\mathbb{S}^n_2 \setminus \{p\}$.

Now suppose that fixed orientations are chosen on the two spheres. Then the restrictions $\varphi|_{B^n_1}, \dots, \varphi|_{B^n_k}$ are called *positive* or *negative* depending on whether they preserve or reverse orientation (i.e., their Jacobians in the chosen coordinate systems are positive or negative). Then we can define the *degree* $\deg(\varphi)$ of the map φ as the difference between the number of positive restrictions (listed above) and the number of negative ones. Intuitively speaking, we can say that $\deg(\varphi)$ is the number of times one sphere wraps around the other.

4.2. Incidence coefficient of two cells

Let $X = \bigcup_i X^{(i)}$ be a CW-space; let $\alpha : \mathbb{D}^{n-1} \to X$ be an $(n-1)$-cell and $\chi = \beta|_{\partial\mathbb{D}^n} : \partial\mathbb{D}^n \to X^{(n-1)}$ be the attaching map of the n-cell $\beta : \mathbb{D}^n \to X^{(n)}$. Denote by $p : X^{(n-1)} \to \mathbb{S}^{n-1}$ the map obtained by compressing to a single point the $(n-2)$-skeleton $X^{(n-2)}$ as well as all the $(n-1)$ cells of $X^{(n-1)}$ except α. Define the map $\widehat{\chi} : \partial\mathbb{D}^n \to \mathbb{S}^{n-1}$ by setting $\widehat{\chi} := p \circ \chi$. By definition, the *incidence coefficient* of the cell β to the cell α is the integer

$$\boxed{[\beta : \alpha] := \deg(\widehat{\chi}) = \deg(p \circ \chi)}.$$

(Here the orientations of the spheres $\partial\mathbb{D}^n$ and $\mathbb{S}^{n-1}$ are those induced by the corresponding cells.) The incidence coefficient is correctly defined if all the maps of the form $\widehat{\chi}$ are neat for all pairs of cells α and β. In that case, we say that the CW-space X is *neat*. We will assume that all CW-spaces considered in the rest of this lecture are neat.

Example. Let X be the CW-space structure on the projective plane $\mathbb{R}P^2$ consisting of three cells of dimensions 0,1,2. The 2-cell $\beta : \mathbb{D}^2 \to X$ is attached to $X^{(1)} = \mathbb{S}^1$ by the map $\chi : \partial\mathbb{D}^2 = \mathbb{S}^1 \to \mathbb{S}^1$, $e^{i\varphi} \mapsto e^{2i\varphi}$. Then the incidence coefficient $[\beta : \alpha]$, which expresses the number of times that the boundary of β wraps around α, is equal to 2.

Remark 1. The key property of neat CW-spaces is that for them the incidence coefficient of two cells is correctly defined via the definition of degree of sphere maps. There are many different ways of ensuring this property, and other authors use, for this purpose, notions other than what I call "neatness".

Remark 2. The general construction described above becomes unrecognizable in dimension $k = 1$, so we will treat this case separately. The 1-disk $\mathbb{D}^1$ is the segment $[0, 1]$, which we always assume oriented from 0 to 1, and any 1-cell σ is determined by a map of its boundary $\partial\mathbb{D}^1 = \{0\} \cup \{1\}$ to the 0-skeleton (the vertices) of our CW-space. Now if the point $\{0\}$ is mapped to some point p of the 0-skeleton, then the corresponding incidence coefficient is set to be equal to -1, while if $\{1\}$ is mapped to some point q, the incidence coefficient taken to be $+1$; in the case $p = q$, the incidence coefficient will be set equal to zero.

4.3. Definition of cellular homology

Suppose that $X = \bigcup_{i=0}^{n} X^{(i)}$ is a finite neat CW-space. Then for each integer k, $0 \le k \le n$, we define the *group of cellular k-chains of X* as the set of all formal linear combinations of k-cells $\gamma_s^k : \mathbb{D}^k \to X^k$,

$$C_k(X) := \Big\{ \sum_{\text{all } k\text{-cells}} z_s \gamma_s^k \;\Big|\; z_s \in \mathbb{Z} \Big\}$$

with integer coefficients z_s, endowed with the natural sum operation.

$$\sum_{\text{all } k\text{-cells}} z_s'\gamma^k + \sum_{\text{all } k\text{-cells}} z_s''\gamma^k = \sum_{\text{all } k\text{-cells}} (z_s' + z_s'')\gamma^k.$$

Under this operation $C_k(X)$ is an Abelian group (actually, it is even a free $\mathbb{Z}$-module). We introduce the *cellular boundary operator* by defining it on each cell by the formula

$$\boxed{\partial_k(\gamma) := \sum_{\text{all } (k-1)\text{-cells}} [\gamma : \beta_m]\,\beta_m}$$

and then extending to the entire group $C_n(X)$ by linearity. This definition makes sense provided we know that the incidence coefficients are well defined; this will be the case if the CW-space X is neat, but we have already assumed this.

Chains $c \in C_k(X)$ such that $\partial_k(c) = 0$ are called *cycles* and those for which there exists a chain $c' \in C_{k+1}$ such that $\partial_{k+1}(c) = c'$ are called *boundaries* (or are said to be *homologous to zero*). Two cycles whose difference is homologous to zero are said to be *homologous*. Being homologous is, of course, an equivalence relation.

We claim that the boundary operator satisfies the *Poincaré Lemma*, i.e., we have

$$\boxed{\partial_{k+1} \circ \partial_k = 0 \quad \text{for all} \quad k \geq 0.}$$

We omit the proof of this statement.

The Poincaré lemma implies that $\mathrm{Im}\partial_{k+1} \subset \mathrm{Ker}\,\partial_k \subset C_k(X)$, and this allows us to define the nth (cellular) *homology group* of a (neat) finite CW-space X by taking the quotient group

$$\boxed{H_k(X) := \frac{\mathrm{Ker}\,\partial_{k+1}}{\mathrm{Im}\,\partial_k} \quad \text{for all} \quad k \geq 0.}$$

Thus elements of the homology group $H_n(X)$ are classes of cycles up to homology.

Cellular homology of finite neat CW-spaces is a functor, so we must define, for every cellular map $f : X \to Y$ and every nonnegative integer n, a homomorphism of $H_n(X)$ to $H_n(Y)$. This is done in the natural way: to every n-cell $\chi : \mathbb{D}^n \to X^{(n)}$ corresponds the cell $f \circ \chi$, and so (by linearity), to each n-chain in X corresponds an n chain in Y; we write

$$f_{*n} : C_n(X) \to C_n(Y);$$

it is not hard to prove that

$$\partial_n^Y \circ f_{*n} = f_{*(n-1)} \circ \partial_n^X,$$

so that cycles correspond to cycles and homologous cycles correspond to homologous cycles (thus the described correspondence is well defined on homology classes), and preserves the sum operation (for details – see the exercise class); the homomorphism $H_n(X) \to H_n(Y)$ thus obtained is denoted by

$$f_* : H_n(X) \to H_n(Y)$$

and called the homomorphism *induced* by f.

Now suppose we are given a pair of (finite neat) CW-spaces (X, A); then a *relative chain* $c \in C_n(X, A)$ is a chain from $C_n(X)$ whose coefficients at cells of A are zero. As above, we define a boundary operator (still denoted by ∂_n),

$$\partial_n : C_n(X, A) \to C_{n-1}(X, A)$$

which satisfies the Poincaré Lemma and so allows us to define, as above, the *relative homology group* $H_n(X, A)$ and $g_* : H_n(X, A) \to H_n(Y, B)$, the homomorphism *induced* by a cellular map $g : (X, A) \to (Y, B)$ of pairs. Note there there is a natural identification $H_n(X, \varnothing) \equiv H_n(X)$.

Remark. Here we defined cellular homology for *finite* CW-spaces because, at this point, we are only interested in the computation of homology groups of compact manifolds, which all have structures of *finite* CW-spaces. Actually the theory in the general case is just the same, it should only be noted that in the definition of an n-chain, one must stipulate that the number of nonzero integer coefficients of each chain be finite.

4.5. Some properties of cellular homology

(i) *Functoriality.* Cellular homology is a functor, which means that

- $(f \circ g)_* = f_* \circ g_*$ for all cellular maps $f : X \to Y$ and $g : Y \to Z$:
- $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$ for any CW-space X and all $n \in \mathbb{N}$.

(ii) *Homology of a point.* $H_0(\text{point}) = \mathbb{Z}$ and $H_n(\text{point}) = 0$ for all $n \geq 1$.

(iii) *Homotopy invariance. Homology groups are homotopy invariant (and therefore topologically invariant)*. In particular, they are independent of the CW-space structure chosen for each CW-space. We omit the proof of this important fact.

(iv) *Zero homology of connected spaces. A CW-space X is path connected if and only if $H_0(X) = \mathbb{Z}$.*

(v) *Exact sequence for pairs.* Given a pair of CW-spaces (X, A), we have the inclusions $i : A \hookrightarrow X$ and $j : X = (X, \varnothing) \hookrightarrow (X, A)$, which induce homomorphisms i_* and j_*. Further, it is not difficult to construct a homomorphism $\partial_* : H_n(X, A) \to H_{n-1}(A)$ (see Exercise 4.11). The three homomorphisms defined above allow to construct a sequence similar to the two sequences for homotopy groups studied in the previous lecture. As before, we have the following statement.

The sequence of homomorphisms

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \dots$$

defined above is exact.

The details concerning items (i), (ii), (iv) (including proofs) will be discussed in the exercise class. At this stage, the proofs of (iii) and (v) are omitted, although I hope that, on the intuitive geometric level, the exactness of the homology sequence for pairs will be understood by everyone.

4.4. Computations and applications

Here we list the values of the homology groups of some of the most popular manifolds. The proofs can be obtained by using the simplest CW-space decompositions of the manifolds (see Lecture 2) and the definition of $H_n(\cdot)$; they will be discussed in the exercise class.

- $H_0(\text{point}) = \mathbb{Z}$, $H_n(\text{point}) = 0$ for $n \geq 1$.
- $H_0(\mathbb{D}^n) = \mathbb{Z}$, $H_k(\mathbb{D}^n) = 0$ for $k \geq 1$.
- $H_0(\mathbb{S}^n) = H_n(\mathbb{S}^n) = \mathbb{Z}$, $H_k(\mathbb{S}^n) = 0$ for all $k \notin \{0, n\}$.
- $H_0(\mathbb{T}^2) = H_2(\mathbb{T}^2) = \mathbb{Z}$, $H_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$, $H_k(\mathbb{T}^2) = 0$ for all $k \geq 3$.
- $H_k(\mathbb{C}P^n) = \mathbb{Z}$ for $k \in \{0, 2, \dots, 2n\}$, $H_k(\mathbb{C}P^n) = 0$ for all other k.
- $H_k(\mathbb{R}P^n) = \mathbb{Z}$ for $k = 0$ and $k = n$ provided n is even, $H_k(\mathbb{R}P^n) = \mathbb{Z}_2$ for odd k less than n, $H_k(\mathbb{R}P^n) = 0$ for all other k.

The values of the homology groups of surfaces (orientable or not, with boundary or not) will be calculated in the exercise class. Unlike homotopy groups, homology groups "behave badly" under the Cartesian product of spaces; however (also unlike homotopy groups), they "behave nicely" under the wedge product, namely:

- $H_n(X \wedge Y) = H_n(X) * H_n(Y)$,

where $*$ denotes the free product of groups.

More facts and other applications will appear in the exercise class.

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4.5. Problems

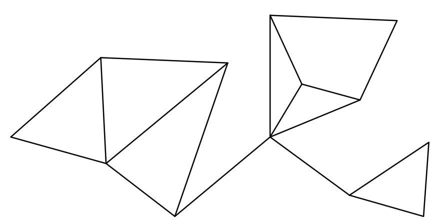
4.1. Compute $H_*(M_g^2)$, where M_g^2 is the sphere with g handles.

4.2. Compute $H_*(N_g^2)$, where N_g^2 is $\mathbb{R}P^2$ with g handles.

4.3. Compute $H_*(\mathbb{C}P^n)$.

4.4. Compute $H_*(\mathbb{R}P^n)$.

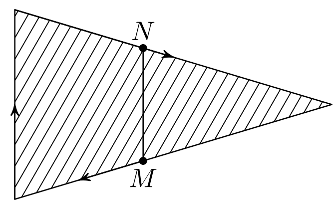
4.5. Compute $H_*(\Gamma)$, where Γ is the following graph.



4.6. Let p and q be two relatively prime positive integers. Consider the action of the group $\mathbb{Z}_p$ with the generator σ on the unit sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ defined by $\sigma(z, w) = \big(\exp\big(2\pi i/p\big)z, \exp\big(2\pi iq/p\big)w\big)$. The quotient of $\mathbb{S}^3$ by this action is a 3-manifold. It is called a *lens space* and is denoted by $L(p, q)$. Compute $H_*(L(p, q)$

4.7. Compute $H_*(\mathcal{D})$, where $\mathcal{D}$ is the dunce hat, i.e., the triangle with the identifications shown by arrows. Can it be retracted to its circle NM?

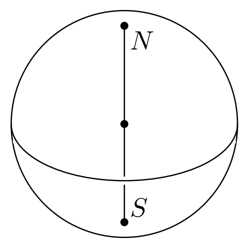
4.8. Can the dunce hat be retracted to its circle NM?



4.9. Express $H_n(A \vee B)$ in terms of $H_n A$ and $H_n B$, $n = 0, 1, \dots$

4.10. Construct the homomorphism $\partial_* : H_n(X, A) \to H_{n-1}(A)$ and show that the homology sequence for pairs is exact.

4.11. Compute $H_*(\bar{S})$, where $\bar{S}$ is the sphere $\mathbb{S}^2$ to which the segment $[NS]$ joining the North and South poles have been added.



Lecture 5

SIMPLICIAL HOMOLOGY

Simplicial homology is the oldest one of the homology theories. It is a functor defined only on the category of simplicial spaces (which is much narrower than the category of topological spaces or even that of CW-spaces); its definition is quite simple and has a rather clear geometric interpretation. However, it is not as convenient for computations as cellular homology (see Lecture 4 and its exercises), and the proof of its main properties is much more difficult than that of the same properties of singular homology (which we will study in Lecture 7).

The construction of the simplicial homology functor (as of other homology functors) is carried out in two basic steps: first, from simplicial spaces and maps (which are geometric entities, see Sect.5.3 below), we pass to chain complexes and morphisms (which are purely algebraic objects), and, second, by a purely algebraic construction, we pass from chain complexes to graded Abelian groups (the homology groups). This second step can be used without any modifications in the construction of other homology theories, e.g. singular homology, Čech homology, cellular homology, etc.

5.1. Chain complexes and their morphisms

A *chain complex* (sometimes also called *graded differential group*) is a sequence of Abelian groups and homomorphisms

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0$$

satisfying the relation $\partial_{n+1} \circ \partial_n = 0$ for all $n = 1, 2, \dots$, or, which is the same thing, the relation $\operatorname{Im} \partial_{n+1} \subset \operatorname{Ker} \partial_n$ for all $n = 1, 2, \dots$. The elements of C_n are called *n-chains* and the homomorphisms ∂_n are called *boundary operators* or *differentials* (their subscript n is sometimes omitted). Elements of $\operatorname{Ker} \partial$ are called *cycles*, elements of $\operatorname{Im} \partial$ are *boundaries*, and two cycles in the same coset modulo $\operatorname{Img} \partial$ are said to be *homologous* or *in the same homology class*. (This terminology, which may seem strange in the abstract algebraic context, actually comes from the geometric aspect of homology theory, where it is quite natural: see Lecture 4.)

Chain complexes form a category whose morphisms $f : \mathcal{C} \to \mathcal{C}'$ are commutative diagrams of the form

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \dots & \xrightarrow{\partial_1} & C_0 \\
 & & \big\downarrow & & \big\downarrow & & & & \big\downarrow \\
\dots & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \xrightarrow{\partial'_{n-1}} & \dots & \xrightarrow{\partial'_1} & C'_0
\end{array}$$

The fact that this is indeed a category (i.e., the two functoriality axioms in the definition of categories, see Lect.1, hold) follows immediately from definitions.

5.2. Homology of chain complexes

The *n-th homology group* of a chain complex $\mathcal{C} = \big(C_n, \partial_n\big)$ is defined as the following quotient group

$$\boxed{H_n(\mathcal{C}) := (\operatorname{Ker} \partial_{n+1})\big/(\operatorname{Im} \partial_n)}$$

Given any morphism $f : \mathcal{C} \to \mathcal{C}'$ of chain complexes, we can construct a homomorphism of the corresponding homology groups

$$f_* : H_n(\mathcal{C}) \to H_n(\mathcal{C}') \quad \text{for all} \quad n \geq 0$$

in the following way. Consider the diagram

$$\begin{array}{ccccc}
C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \\
\big\downarrow{\scriptstyle f_{n+1}} & & \big\downarrow{\scriptstyle f_n} & & \big\downarrow{\scriptstyle f_{n-1}} \\
C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1},
\end{array}$$

take some element $c \in \operatorname{Ker}(\partial_n)$, and let h be its homology class. Define $c' := f_n(c)$, and denote by h' the homology class of c'. Now we can define the *induced homomorphism* $f_* : H_n(\mathcal{C}) \to H_n(\mathcal{C}')$ by assigning $h \mapsto h'$. The fact that f_* (here and elsewhere we omit the subscipt n) is a well-defined homomorphism follows by an fairly easy chase in the last diagram.

It follows directly from the definitions that *the assignment described above of homology groups $H_*(\mathcal{C})$ and induced homomorphisms f_* to chain complexes $\mathcal{C}$ and their morphisms f is a functor, i.e.,*

$$\big(f \circ g\big)_* = f_* \circ g_* \quad \text{and} \quad (\mathrm{id}_{\mathcal{C}})_* = \mathrm{id}_{H_*(\mathcal{C})} \,.$$

5.3. Simplicial spaces

An *n-dimensional simplex* (*n-simplex* for short) Δ^n is the convex hull of $n+1$ points (called *vertices*) in $\mathbb{R}^n$, namely the origin and the endpoints of the basis vectors; Δ^n is supplied with the induced topology. Thus the 0-simplex is the point, the 1-simplex is the closed interval $\mathbb{I} = [0, 1]$, the 2-simplex is the triangle, the 3-simplex is the tetrahedron, etc. For convenience, we will regard the empty set as a (-1)-dimensional simplex. Note that we regard the n-simplex as a topological space (homeomorphic to the disk $\mathbb{D}^n$), but with an *additional structure*: the collection of its k-faces, $k = 0, \dots, n-1$, defined as follows. The 0-*faces* are the vertices, each of the *k-faces*, $1 \le k \le n$, is obtained by choosing $k+1$ vertices of Δ^n and taking their convex hull. Note that each k-face has the structure of a k-simplex.

Roughly speaking, a simplicial space is a topological space glued together from a collection of simplices according to certain rules. It is a very particular case of a CW-space: the gluing rules for simplices are much more restrictive than those for cells. More precisely, a *simplicial space* is defined as a topological space X presented as the union of a family of simplices,

$$X = \bigcup_{(\alpha,i)} \sigma^i_\alpha \quad \sigma^i_\alpha \approx \Delta^i,$$

and satisfying the three following conditions:

- (*triangulation*) any two simplices intersect in a common face (the empty set is regarded as a face of each simplex);
- (*completeness*) any face of any simplex is a simplex of X;
- (*local finiteness*) any face is the face of only a finite number of simplices.

Simplicial spaces form a category if for morphisms we take *simplicial maps*, i.e., continuous maps $f : X \to Y$ that take any simplex $\sigma^k \subset X$ linearly onto a simplex $\sigma^l \subset Y$, where $l \le k$.

According to the definition, any simplicial space X is supplied with a combinatorial structure, i.e, it has a fixed decomposition into simplices. It is sometimes useful to change this structure by subdividing the simplices of X into smaller ones. In Figure 1 we show several ways of subdividing a 2-simplex Δ; the last of these is called the *baricentric subdivision* of Δ and is obtained by subdividing each side (1-face) of Δ into two 1-simplices by means of the midpoint of the side and then taking the cones with vertex g

(the baricenter of Δ) over the 6 obtained 1-simplices, thus obtaining 6 new 2-simplices.

The definition of the *baricentric subdivision of an n-simplex* Δ^n is similar: we take the baricentric subdivision of all the faces of Δ^n and construct the cones over them with vertex g (the baricenter of Δ^n). By the *baricentric subdivision of a simplicial space* X we mean the result X' of the simultaneous baricentric subdivision of all the simplices of X. By iterating this procedure, we can obtain the simplicial space $X^{(n)}$ consisting of the same points as X but with simplices as small as we wish.

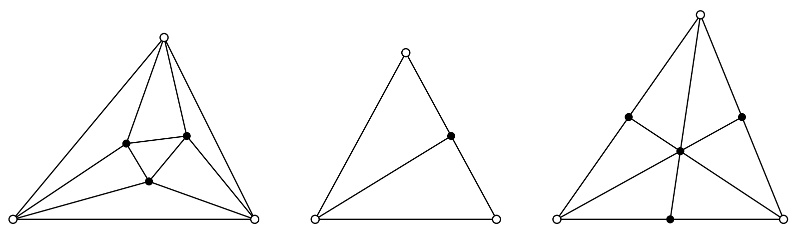


FIGURE 5.1. Subdivisions of a 2-simplex

Simplicial spaces have a topological structure and can be treated as topological spaces. The connection between the two approaches is given by the following important theorem.

Theorem (Simplicial Approximation.) *Any continuous map $f : X \to Y$ of simplicial spaces is homotopic to a simplicial map $s : X^{(n)} \to Y^{(n)}$ of the nth baricentric subdivisions of X and Y for some n. The map s may be chosen as close to f as we wish.*

The proof of this theorem is somewhat simpler than that of the cellular approximation theorem, but uses other tools. It is omitted in this course.

Besides its topological and combinatorial structure, a simplicial space (which is an abstract entity) can be endowed with a very concrete geometric structure. Indeed, we have the following

Theorem (Polyhedral Realization.) *Any abstract simplicial space X of dimension n can be realized as a polyhedron $P(X)$ in Euclidean space $\mathbb{R}^{2n+1}$ decomposed into rectilinear simplices that bijectively correspond to the simplices of X, with faces corresponding to faces.*

This theorem is not needed further in this course, so its (not very difficult) proof is omitted.

5.4. The chain complex of a simplicial space

We shall now define the chain complex $C_*(X)$ of an arbitrary simplicial space X. The main protagonists in our chain complex will be *oriented simplices*, i.e., simplices with a fixed orientation. To define the orientation of a simplex, we write out the sequence of its vertices $v_0, v_1, \dots, v_n$ in some order; there are $n!$ such orders, and we call two orders *equivalent* if one can be obtained from the other by an even number of transpositions; for a fixed simplex, there are obviously two equivalence classes of orderings of its vertices; each of these classes is said to be an *orientation* of the simplex: the oriented n-dimensional simplex determined by the ordering $v_0, v_1, \dots, v_n$ of its vertices, i.e., the equivalence class containing this order, will be denoted $\sigma^n = [v_0, v_1, \dots, v_n]$. Geometrically, the orientation of a 1-simplex is shown by an arrow going from one of its vertices to the other, the orientation of a 2-simplex is a direction of rotation of the plane in which the simplex lies. As to zero-dimensional simplices (i.e., points), let us agree that the orientation is simply the assignment of a plus sign or a minus sign to the 0-simplex. We will also agree to denote by $-[v_0, v_1, \dots, v_n]$ the simplex with the same vertices, but with the orientation opposite to that of the simplex $[v_0, v_1, \dots, v_n]$.

Remark. Physicists distinguish positive and negative orientations, e.g. in the plane, they consider the counterclockwise rotation positive; in 3-space they talk about right-hand screws (which determine the positive orientation of a 3-simplex), there is "left-hand rule" in electromagnetism and so on. None of these distinctions make any sense mathematically, and so we will not appeal to these mathematically meaningless preferences in the choice of orientation.

By definition, an *n-dimensional chain c* is a linear combination with integer coefficients (only a finite number of which are nonzero) of all oriented n-simplices from X,

$$c = \sum_i z_i \sigma_i^n \in C_n(X).$$

The set $C_n(X)$ of all n-chains, called the *nth chain group* has an obvious structure of an Abelian group generated by all the (ordered) n-simplices (actually it is a free $\mathbb{Z}$-module); the direct sum $\bigoplus_{n \geq 0} C_n(X)$ of the n-chain groups is also an Abelian group denoted $C_*(X)$.

To obtain a chain complex from the groups $C_n(X)$, we must define the *boundary operators* or *differentials*

$$\partial_n : C_n(X) \to C_{n-1}(X).$$

If $\sigma^n = [v_0, v_1, \dots, v_n]$ is an (oriented) n-simplex, we write

$$[v_0, \dots, v_j^\vee, \dots, v_n] := [v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_n]$$

for its $(n-1)$-face obtained by throwing out its jth vertex. We now define the boundary operator by setting

$$\partial_n c = \partial_n\Big(\sum_{i\geq 1} z_i \sigma_i^n\Big) := \sum_{i\geq 1} z_i \partial_n(\sigma_i^n) := \sum_{i\geq 1} z_i \sum_{j=0}^{n} (-1)^j [v_0, \dots, v_j^\vee, \dots, v_n].$$

The fact that ∂_n is well defined is considered in Exercise 5.6.

Lemma (Poincaré).

$$\boxed{\partial_{n-1} \circ \partial_n = 0 \quad \text{for any } n \geq 2.}$$

Proof. Because of the linearity of ∂_n, it suffices to prove the lemma for the case in which the chain c contains only one nonzero coefficient z, and this coefficient equals 1, so that $c = \sigma$, where σ is some n-simplex from X. But in this case it is obvious, because (by the definition of the boundary operators) the $(n-2)$-chain $\big(\partial_{n-1} \circ \partial_n\big)(\sigma)$ consists of $2n$ summands, which occur in pairs of the form

$$[v_0, \dots, v_j^\vee, \dots, v_k^\vee, \dots, v_n].$$

with opposite signs, so that the whole sum cancels out. □

Remark. Actually three different types of simplices can be used to define simplicial homology: oriented simplices (as was done above), ordered simplices, obtained by numbering all the vertices of the simplicial space and assigning to each geometric simplex the unique ordered simplex whose vertices occur in ascending order, and finally ordered simplices (then each geometric n-simplex determines $n!$ different ordered simplices, which are the generators of the group of n-chains). Thus we obtain three different chain complexes, but it turns out that the resulting homology theories coincide (see Exercise 5.11 and Section 10.3 in the lecture on Poincaré duality).

Now let us define the *morphism of chain complexes* corresponding to a simplicial map $f : X \to Y$; this is done in the natural way, i.e., given a chain $c \in C_k(X)$ and a simplex σ_k appearing in it with nonzero coefficient, we take its image $f(\sigma^k) = \tau^l$ with the same coefficient as a summand of the image chain, while if $l < k$, we do nothing (adding, so to speak, the zero summand to the image chain); we do this for all simplices appearing in c with nonzero coefficients and then combine (add together) the coefficients at like simplices $\tau_j^k \subset Y$, obtaining the image chain, denoted by $f_{*k}(c) \in C_k(Y)$.

It is easy to verify that

$$\partial_n^Y \circ f_{*n} = f_{*(n-1)} \circ \partial_n^X,$$

so that morphisms of chain complexes correspond to simplicial maps, and the above constructions define a functor from the category of simplicial spaces $\mathcal{S}$im to the category of chain complexes $\mathcal{CC}$.

Once this is done, the definitions of the *simplicial homology groups* and their *induced homomorphisms*

$$H_*(X) = \bigoplus_{n\geq 0} H_n(X), \quad f_* : H_n(X) \to H_n(Y)$$

is immediate: one simply takes (see Sec.5.2) the homology groups and the induced homomorphisms of the chain complex constructed above.

5.5. Relative homology of simplicial spaces

A *pair* (X, A) of simplicial spaces is simply a simplicial space X with a subset A consisting of simplices of X such that A inherits its own structure of a simplicial space (i.e., the three conditions in the definition of such spaces hold, see Sect.5.3). Simplicial maps of pairs of spaces are defined in the natural way.

Given a simplicial pair (X, A) and a nonnegative integer n, we define the *relative chain group* $C_n(X, A)$ as the subgroup of $C_n(X)$ all of whose chains have zero coefficients at n-simplices from A. Further, one can define the boundary operators $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$ in the natural way (so that the ordinary boundary of a chain $c \in C_n(X, A)$ that lies in $C_{n-1}(A)$ is regarded as zero) and go on to the definition of the *relative homology groups* and the homomorphisms *induced* by simplicial maps,

$$H_*(X, A) = \bigoplus_{n\geq 0} H_n(X, A), \quad f_* : H_n(X, A) \to H_n(Y, B).$$

We omit the details at this stage and note without proof that we have defined a functor from the category of pairs of simplicial spaces to the category of graded groups.

5.6. Homology with arbitrary coefficients

When we defined the chain groups $C_n(X)$ and $C_n(X, A)$, we used elements of the group of integers $\mathbb{Z}$ for the coefficients appearing in the chains. Actually, instead of $\mathbb{Z}$, one can take any other Abelian group $\mathbb{G}$ (e.g. the additive group of any field, or residues $\bmod m$) and continue further constructions exactly in the same way. One then obtains the homology groups $H_n(X; \mathbb{G})$ and $H_n(X, A; \mathbb{G})$, called *homology groups modulo* $\mathbb{G}$, with similar induced homomorphisms and boundary operators. The theory is exactly the same as for the *integer homology groups* $H_n(X; \mathbb{Z}) \equiv H_n(X)$, but for certain applications, the homology groups $H_n(X; \mathbb{R})$ or $H_n(X; \mathbb{Z}_2)$ are more convenient and efficient than $H_n(X; \mathbb{Z})$.

5.7. Zero homology and augmentation

The definition of homology given in this lecture and the previous one is perhaps unclear in the case $n = 0$, because the boundary operator ∂_n was not specified for $n = 0$. If we define it as taking everything to zero, $\partial_0 : C_0(X) \longrightarrow 0$, then we should define the corresponding zero homology by setting $H_0(X) := C_0(X)/\operatorname{Im} \partial_0$.

According to this definition, it is obvious that *the simplicial space X is connected if and only if $H_0(X) \cong \mathbb{Z}$.*

In order to simplify certain formulations, it is convenient to slightly modify the very end of the chain complex corresponding to a given simplicial space, replacing the last two terms $\ldots \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow 0$ by the following three-term sequence

$$\ldots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} C_{-1}(X) = \mathbb{Z} \longrightarrow 0,$$

where the homomorphism ∂_0 is defined by the formula

$$\partial_0\Big(\sum z_i v_i\Big) = \sum z_i.$$

Then we can define zero homology (now denoted by $\widetilde{H}_0$) on the standard way by setting $\widetilde{H}_0(X) := \operatorname{Ker} \partial_1/\operatorname{Im} \partial_0$. Then *the simplicial space X is connected if and only if $\widetilde{H}_0(X) = 0$.* It is also easy to see that we always have $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z}$.

5.8. Problems

5.1. Consider the chain complex

$$0 \xrightarrow{\partial_4} \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} 0$$

where $\partial_3(z) = 2z$ and $\partial_2(z) = 0$. Compute its homology.

5.2. Compute the homology of the singleton and the segment $[0, 1]$ regarded as a simplicial space with one 1-simplex.

5.3. Compute the homology of the subdivided segment (see the figure) by using only the definition of homology groups and then verify your answer by using their properties (such as homotopy invariance).

5.4. Compute the homology of the boundary of the triangle and the boundary of the square (by using the definition of simplicial homology groups). Compare.

5.5. Compute the homology of the boundary of the tetrahedron and the boundary of the cube (its faces are triangulated by means of their diagonals). Compare.

5.6. Prove that the induced homomorphism in simplicial homology theory is well defined, i.e., does not depend on the representative of the orientation.

5.7. Prove the Poincaré lemma in detail (check that the two simplices $[v_0, \ldots, \breve{v}_j, \ldots, \breve{v}_i, \ldots, v_n]$ do appear with opposite signs).

5.8. Prove that a simplicial space X is connected if and only if $H_0 X = \mathbb{Z}$.

5.9. Compute the homology groups $H_*(M\ddot{o}b, \mathbb{Z})$ of the Möbius strip directly from the definition of simplicial homology.

5.10. Compute the homology of $\mathbb{R}P^2$ modulo 2 (i.e., with coefficients in $\mathbb{Z}_2$).

5.11. Define the homology theory for ordered simplices (in which each geometric n-simplex yields $n!$ ordered simplices) and prove that it is equivalent to the theory involving all oriented simplices and to the *ordered theory*, in which the vertices of the simplicial space are ordered (so that there is only one ordered simplex corresponding to each geometric simplex).

Lecture 6

PROPERTIES OF SIMPLICIAL HOMOLOGY

The aim of this lecture is to establish some basic properties of the simplicial homology groups. We begin, however, with some algebraic preliminaries mostly related to chain complexes (short exact sequences of chain complexes and the corresponding long homology sequences, some auxiliary statements such as Steenrod's five-lemma, chain homotopy, etc.). We also discuss the notion of acyclic support, a useful geometric tool that yields important information in algebraic topology, e.g. in constructing chain homotopies.

After this is done, it turns out that a series of fundamental results are obtained without much extra work. They are the homotopy invariance (and hence the topological invariance) of homology groups, the exact homology sequence for pairs, the Hurewicz theorem (which establishes a fundamental relationship between homology and homotopy groups), the Mayer–Vietoris sequence (which often allows to compute the homology of a space when we know the homology of its parts).

6.1. Four algebraic lemmas

In the previous lecture, we already mentioned that a sequence of groups and homomorphisms

$$\dots \xrightarrow{h_{i-1}} G_{i-1} \xrightarrow{h_i} G_i \xrightarrow{h_{i+1}} G_{i+1} \xrightarrow{h_{i+2}} \dots$$

is called *exact at the term* G_i if we have $\operatorname{Im} h_i = \operatorname{Ker} h_{i+1}$; the whole sequence is called *exact* if it is exact at all terms.

The following properties of exact sequences of groups, which immediately follow from their definition, hold:

(i) *if the sequence* $0 \to A \to B \to \dots$ *is exact, then the homomorphism* $A \to B$ *is a monomorphism;*

(ii) *if the sequence* $\dots \to A \to B \to 0$ *is exact, then the homomorphism* $A \to B$ *is an epimorphism;*

(iii) *if the sequence* $0 \to A \to B \to 0$ *is exact, then the homomorphism* $A \to B$ *is an isomorphism.*

A five-term exact sequence $0 \to A \to B \to C \to 0$ starting and ending in zero is called *short exact*.

Lemma (Short Exact Lemma). *If $h : G \to H$ is an arbitrary group homomorphism, then*

$$0 \longrightarrow \operatorname{Ker} h \xrightarrow{\ i\ } G \xrightarrow{\ h\ } \operatorname{Im} h \longrightarrow 0,$$

where i is the inclusion homomorphism, is a short exact sequence.

The proof of this lemma is obvious.

Lemma (Splitting Lemma.) *The condition that the short exact sequence*

$$0 \longrightarrow A \xrightarrow{\ \varphi\ } B \xrightarrow{\ \psi\ } C \longrightarrow 0,$$

splits, i.e., can be rewritten in the form

$$0 \longrightarrow A \xrightarrow{\ i\ } A \oplus C \xrightarrow{\ p\ } C \longrightarrow 0,$$

where i is the natural inclusion and p the projection on the second factor, is equivalent to the condition that φ has a left inverse (i.e., there exists a homomorphism $\Phi : B \to A$ such that $\varphi \circ \Phi = \mathrm{id}_B$) as well as to the condition ψ has a right inverse (i.e., there exists a homomorphism $\Psi : C \to B$ such that $\Psi \circ \psi = \mathrm{id}_B$).

The proof of this lemma is an (easy) exercise.

The next algebraic lemma is used in homotopy and homology theories to prove that different spaces have isomorphic homotopy (homology) groups. A few instances of its application will appear in the exercise class.

Lemma (Steenrod's Five-Lemma). *In the commutative diagram*

$$\begin{array}{ccccccccc} A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C & \xrightarrow{\ h\ } & D & \xrightarrow{\ i\ } & E \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\ A' & \xrightarrow{\ f'\ } & B' & \xrightarrow{\ g'\ } & C' & \xrightarrow{\ h'\ } & D' & \xrightarrow{\ i'\ } & E' \end{array}$$

let the rows be exact and the vertical homomorphisms p, q, s, t be isomorphisms. Then r, the middle vertical arrow, is also an isomorphism.

The proof is a nice exercise about abstract groups and diagram chasing. In fact, the assertion of this lemma holds under weaker conditions. This will be discussed in the exercise class.

Lemma ((3 × 3)-Lemma). *Suppose that in the following commutative diagram*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p''} & & \\
0 & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & 0 \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q''} & & \\
0 & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{i'} & 0 \\
 & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle r''} & & \\
0 & \xrightarrow{f''} & B'' & \xrightarrow{g''} & C'' & \xrightarrow{h''} & D'' & \xrightarrow{i''} & 0 \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle s''} & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

the first and second (*second and third*) *rows and all the columns are exact; then so is the third* (*first*) *row.*

The proof is another typical simple example of diagram chasing.

6.2. Constructing long homology sequences.

We have already constructed long exact homotopy sequences in the previous lectures by *ad hoc* methods. The next lemma is the main algebraic tool for constructing long homology sequences of different types in a more "scientific way".

Lemma (Short to Long Exact Lemma). *If the short sequence of chain complexes* $0 \to \mathcal{C} \to \mathcal{C}' \to \mathcal{C}'' \to 0$*, i.e,*

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_i & \longrightarrow & C'_i & \xrightarrow[j]{} & C''_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_i} & & \downarrow{\scriptstyle \partial'_i} & & \downarrow{\scriptstyle \partial''_i} & & \\
0 & \longrightarrow & C_{i-1} & \longrightarrow & C'_{i-1} & \xrightarrow[j]{} & C''_{i-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

is exact, then it induces the following long exact sequence in homology:

$$\cdots \to H_i(\mathcal{C}) \to H_i(\mathcal{C}') \to H_i(\mathcal{C}'') \to H_i(\mathcal{C}) \to \cdots \to H_0(\mathcal{C}'').$$

Proof. Let us construct the homomorphism $H_i(\mathcal{C}'') \to H_{i-1}(\mathcal{C})$. Pick some $c_i'' \in \operatorname{Ker} \partial_i''$. It follows from the exactness of the horizontal sequences that $c_i'' = j(c_i')$ for some c_i', and there is an element c_{i-1} in C_{i-1} which is the preimage of $\partial_i'(c_i')$. The element c is a cycle in the complex $\mathcal{C}$, and we take the class corresponding to c in the group $H_{i-1}(\mathcal{C})$ to be the image of the homology class of the cycle c_i''. Thus we have constructed a map from $H_i(\mathcal{C}'')$ to $H_{i-1}(\mathcal{C})$. It is easy to check that this map is a well-defined homomorphism. The construction of the other homomorphisms in the long sequence and the verification of its exactness is straightforward diagram chasing.

6.2. Chain homotopy

Two morphisms $\{f_k\}$ and $\{g_k\}$ between two chain complexes $\{C_k'\}$ and $\{C_k''\}$ are called *chain homotopic* if there exists a family of homomorphisms $D_k : C_k' \to C_{k+1}''$ such that

$$\partial_{k+1}'' \circ D_k + D_{k-1} \circ \partial_k' = g_k - f_k;$$

the family $\{D_k\}$ is then called a *chain homotopy* between f and g.

Remark. This definition may seem rather strange at first glance. In order to understand what it means, the reader should consider the case in which the chain complexes are simplicial ones and try to give a geometric interpretation of the homomorphisms D_k.

Lemma (Chain Homotopy). *If the morphisms $\{f_k\}$ and $\{g_k\}$ between two chain complexes $\{C_k'\}$ and $\{C_k''\}$ are chain homotopic, then the corresponding induced homomorphisms in homology $\{(f_k)_*\}$ and $\{(g_k)_*\}$ coincide.*

Proof. Let $z_k \in C_k'$ be a cycle, i.e., $\partial_k'(z_k) = 0$. Then

$$g_k(z_k) - f_k(z_k) = \partial_{k+1}''(D_k(z_k)) + D_{k-1}(\partial_k'(z_k)) = \partial_{k+1}''(D_k z_k),$$

which means that $g_k(z_k)$ and $f_k(z_k)$ are homologous. □

6.3. Acyclic supports

We now move away from the purely algebraic context of chain complexes back to simplicial spaces. A simplicial space is said to be *acyclic* if its homology is zero in all dimensions $n > 0$ and $H_0(X) = \mathbb{Z}$. The *support* of a chain

$c \in C_n(X)$ is any simplicial subspace of X that contains all the simplices which appear in the chain with nonzero coefficients.

For the statement of the next lemma, we will need a technical notion related to 0-dimensional chains: a chain map f is called *augmentation-preserving* if

$$f_0\Big(\sum_i a_i \Delta_i^0\Big) = \sum_i b_j \Delta_j^0 \quad \text{with} \quad \sum_i a_i = \sum_i b_j.$$

(The motivation for introducing the notion of augmentation was discussed in the previous lecture.)

Lemma (Acyclic Support Lemma). *Suppose X and Y are simplicial spaces and two augmentation-preserving chain maps*

$$\varphi_k, \psi_k : C_k(X) \to C_k(Y)$$

are given. Assume that an assignment A taking each simplex $\Delta \subset X$ to a simplicial subspace $A(\Delta) \subset Y$ is given and it satisfies the conditions

(i) $\Delta' \subset \Delta$ *implies* $A(\Delta') \subset A(\Delta)$*;*

(ii) $A(\Delta)$ *is acyclic;*

(iii) *the set* $A(\Delta^k)$ *is the support of both chains* $\varphi_k(\Delta^k)$ *and* $\psi_k(\Delta^k)$.

Then the maps φ_k and ψ_k are chain homotopic and so $\varphi_* = \psi_*$.

Proof. We will construct a chain homotopy $D_k : C_k(X) \to C_{k+1}(Y)$ by induction on k. We begin with the case $k = 0$. Let Δ^0 be a vertex of X. The simplicial space $A(\Delta^0)$ supports both chains $\varphi_0(\Delta^0)$ and $\psi_0(\Delta^0)$. Since the maps φ_0 and ψ_0 preserve augmentation, we have

$$(\varphi_0 - \psi_0)(a\Delta^0) = \sum b_i \Delta_i^0, \quad \text{where} \quad \sum b_i = a.$$

Since $A(\Delta^0)$ is acyclic, the chain $\sum b_i \Delta_i^0$ must be the boundary of some 1-chain (which we denote $D_0(a\Delta^0)$) lying in $A(\Delta^0)$, i.e.,

$$(\varphi_0 - \psi_0)(a\Delta^0) = \partial_1 D_0(a\Delta^0).$$

However, we don't necessarily have the relation

$$D_0(a\Delta^0) + D_0(b\Delta^0) = D_0((a+b)\Delta^0),$$

but it will hold if we redefine D_0 by setting

$$D_0(a\Delta^0) = aD_0(1 \cdot \Delta^0).$$

(Further on, chains of the form $1 \cdot \Delta$ will be simply denoted by Δ).

To perform the induction step, suppose that the required homomorphisms $D_0, D_1, \dots, D_{k-1}$ have been constructed, and $A(\Delta^i)$ supports the chain $D_i(\Delta^i)$. We must construct a homomorphism $D_k : C_k(X) \to C_{k+1}(Y)$ which satisfies the only condition that for any k-dimensional simplex $\Delta \subset X$,

$$\partial_{k+1} D_k(\Delta) = c_k, \quad \text{where} \quad c_k = \varphi_k(\Delta) - \psi_k(\Delta) - D_{k-1}\partial_k(\Delta).$$

Now all the simplices of $\partial_k(\Delta)$ are contained in Δ, and therefore $A(\Delta)$ supports the chain $\partial_k(\Delta)$ and so supports the chain $D_{k-1}\partial_k(\Delta)$. Thus $A(\Delta)$ supports the chain c_k and the induction hypothesis implies that

$$c_k = (\psi_k - \varphi_k - D_{k-1}\partial_k)(\Delta) = \\ = (\psi_k - \varphi_k - (\psi_{k-1}\partial_k - \varphi_{k-1}\partial_k - D_{k-2}\partial_{k-1}\partial_k))(\Delta).$$

The support $A(\Delta)$ is acyclic, therefore the cycle c_k is the boundary of some chain (which we denote $D_k(\Delta)$) supported by $A(\Delta)$ and satisfying the equality $\partial_{k+1} D_k(\Delta) = c_k$, as required. □

6.4. Homotopy invariance of homology

A simplicial space is, of course, a topological space, and as such we can speak of its homotopy equivalence to some other space. We will not prove that simplicial homology theory is a homotopy invariant (in this general topological sense). It is more in the spirit of the category approach to mathematics to give a more combinatorial definition of homotopy and homotopy equivalence for simplicial spaces and then prove that homology is homotopy invariant in the combinatorial sense (often also called piecewise linear). This is the goal of this section.

A continuous map $f : X \to Y$ of simplicial spaces is called *piecewise linear* (briefly, *PL*) if it is simplicial in some subdivision of the simplicial structures of X and Y. Two PL maps $f, g : X \to Y$ are called *PL-homotopic* if there exists a PL-map $F : X \times [0,1] \to Y$ such that

$$F(x,0) = f(x), F(x,0) = g(x) \quad \text{for all} \quad x \in X;$$

here the Cartesian product $X \times [0,1]$ is supplied with a PL-structure in the natural way (in particular, the simplicial structure on $X \times \{0\}$ and $X \times \{1\}$ is the same as that on X and for any simplex $\Delta \subset X$ the set $\Delta \times \mathbb{I}$ is a simplicial subset of $X \times [0,1]$).

Theorem. *PL-homotopic PL-maps $f, g : X \to Y$ of simplicial spaces induce the same homomorphism in homology.*

Proof. Let F denote a homotopy between f and g. Consider the inclusions

$$i_0 : X \hookrightarrow X \times \{0\} \subset X \times \mathbb{I} \quad \text{and} \quad i_1 : X \hookrightarrow X \times \{1\} \subset X \times \mathbb{I}.$$

We obviously have $f = F\,i_0$ and $g = F\,i_1$, hence it suffices to prove that $i_{0*} = i_{1*}$. Let Δ^k be a simplex in X. The chains $i_0(\Delta^k)$ and $i_1(\Delta^k)$ have the same support $\Delta^k \times \mathbb{I}$, which is obviously acyclic. By the Acyclic Support Lemma, this means that there is chain homotopy joining i_{0*} and i_{1*}, and so by the Chain Homotopy Lemma $i_{0*} = i_{1*}$. □

Two simplicial spaces X and Y are called *PL-homotopy equivalent* if there exist simplicial subdivisions of X and Y and simplicial maps (w.r.t. these subdivisions) $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are PL-homotopic to the identity maps of Y and X, respectively. The following corollary of the previous theorem is a direct consequence of the functoriality of simplicial homology.

Corollary. *Simplicial homology groups are invariants of PL-homotopy equivalence.*

Two simplicial spaces X and Y are called *PL-homeomorphic* or *PL-equivalent* if there exist simplicial subdivisions of X and Y and a simplicial (w.r.t. these subdivisions) homeomorphism of X onto Y. The fact that PL-equivalence (obviously) implies homotopy equivalence yields the next corollary.

Corollary. *Simplicial homology groups are invariants of PL-equivalence.*

Remark. At this stage, with the use of the simplicial approximation theorem and some extra technical efforts (not involving any new ideas), it is possible to prove the (stronger) purely topological versions of the above theorem and its corollaries. We omit these proofs mainly for aesthetic reasons, referring the reader to V.Prasolov's book *Elements of Homology Theory*, Chap.1, Sect. 2. (Part of the above exposition follows that book rather closely, including the notation.)

6.5. Exact homology sequence for pairs

For the homology groups of pairs of simplicial spaces, we have an exact sequence very similar to the one for homotopy groups.

Theorem (Homology sequence for pairs). *For any simplicial pair* (X, A), *we have the following exact sequence:*

$$\dots \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X)$$
$$\xrightarrow{j} H_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{i} H_0(X) \xrightarrow{j} H_0(X,A).$$

Proof. Given a simplicial pair (X, A), consider the three chain complexes $\mathcal{C}(A)$, $\mathcal{C}(X)$, $\mathcal{C}(X, A)$. There is a morphism of the first complex to the second one (inclusion), and of the second one to the third (factorization), yielding the sequence of chain complexes $0 \to \mathcal{C}(A) \to \mathcal{C}(X) \to \mathcal{C}(X,A) \to 0$, which can be written in more detail as

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \xrightarrow[j]{} & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_*} & & \downarrow{\scriptstyle \partial_*} & & \downarrow{\scriptstyle \partial_*} & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \xrightarrow[j]{} & C'_{n-1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

But this is a short exact sequence of chain complexes, and it gives the required long exact sequence by the Short to Long Exact Lemma. $\square$

6.6. Mayer–Vietoris sequence

The exact sequence in question, known as the *Mayer–Vietoris sequence*, although it was actually first discovered by M.F.Bokstein, is a relationship between the homology of the union of two spaces and the homology of each of the spaces and that of their intersection.

Theorem (Mayer–Vietoris). *Let* X_1 *and* X_2 *be simplicial subspaces of the simplicial space* X *such that* $X = X_1 \cup X_2$*; denote* $Y = X_1 \cap X_2$*. Then we have the following exact sequence:*

$$\dots \xrightarrow{\partial_*} H_n(Y) \xrightarrow{i_*} H_n(X_1) \oplus H_n(X_2) \xrightarrow{j_*} H_n(X) \xrightarrow{\partial_*} \dots,$$

Proof. We obviously have

$$\mathcal{C}(Y) = \mathcal{C}(X_1) \cap \mathcal{C}(X_2) \quad \text{and} \quad \mathcal{C}(X_1) + \mathcal{C}(X_2) = \mathcal{C}(X_1) \cup \mathcal{C}(X_2)$$

We have the inclusion maps $i_1 : Y \hookrightarrow X_1$ and $i_2 : Y \hookrightarrow X_2$, as well as $j_1 : X_1 \hookrightarrow X$ and $j_2 : X_2 \hookrightarrow X$. For $c \in \mathcal{C}(Y)$ put $i(c) := (i_1(c), -i_2(c))$, and put $j(c_1, c_2) = j_1(c_1) + j_2(c_2)$ for $c_1 \in \mathcal{C}(X_1)$ and $c_2 \in \mathcal{C}(X_2)$. Thus we obtain the following short sequence of chain complexes

$$0 \longrightarrow \mathcal{C}(X_1 \cap X_2) \xrightarrow{\ i\ } \mathcal{C}(X_1) \oplus \mathcal{C}(X_2) \xrightarrow{\ j\ } \mathcal{C}(X_1 \cup X_2) \longrightarrow 0.$$

Now it is easy to see that this short sequence is exact. Using the Short to Long Exact Lemma, we obtain a long homology sequence, which is exactly the one appearing in the statement of the theorem. □

6.7. Hurewicz theorem

The Hurewicz theorem describes an important relationship between the n-dimensional homology and homotopy groups. For $n = 1$, it was actually first discovered by Poincaré, and asserts that $H_1(X, \mathbb{Z})$ is simply the abelianization of the fundamental group $\pi_1(X, p)$.

Let us denote the commutator subgroup of π_1 by

$$[\pi_1(X,p), \pi_1(X,p)] := \{aba^{-1}b^{-1} \mid a, b \in \pi_1(X,p)\}.$$

Theorem (Poincaré–Hurewicz). *The first homology group $H_1(X, \mathbb{Z})$ of a connected simplicial space X is isomorphic to the abelianization*

$$\pi_1(X,p)/[\pi_1(X,p), \pi_1(X,p)]$$

of the fundamental group $\pi_1(X,p)$ of X for any basepoint p.

Proof. To each element $\alpha \in \pi_1(X,p)$ let us assign an element of $H_1(X)$. By the simplicial approximation theorem, there is a simplicial path $a \in \alpha$ (in some triangulation of X). Denote by $h(\alpha)$ the oriented chain obtained by assigning the coefficient $+1$ to each oriented 1–simplex of the path a and 0 to all the other 1-simplices. Now assign the homology class of $h(a)$ to the element $\alpha \in \pi_1(X,p)$. It is not difficult to check that if we replace α by another element of the coset modulo the commutator subgroup of π_1, we obtain the same 1-homology class. Moreover, it turns out that this assignement is an isomorphism of the abelianization of $\pi_1(X)$ onto $H_1(X)$. □

This isomorphism is called the *Hurewicz isomorphism.* It is a particular case of the Hurewicz homomorphism in dimension n ($n \geq 1$), which is defined as follows.

Let $\sigma : \mathbb{S}^n \to X$ be a spheroid in a path connected simplicial space X with basepoint $x_0 \in X$. Then, by the simplicial approximation theorem, the map σ can be approximated by a simplicial map $\bar{\sigma}$, which determines the same element of $\pi_n(X, x_0)$. Consider the chain $c_\sigma \in C_n(X; \mathbb{Z})$ in which all the n-simplices of the image $\bar{\sigma}(\mathbb{S}^n)$ appear with coefficient 1, while all the other simplices, with coefficient 0. It is not hard to verify that this chain is a cycle that determines a well-defined homomorphism $\gamma : \pi_n(X, x_0) \to H_n(X, \mathbb{Z})$; the assignment γ is known as the *Hurewicz homomorphism.*

Theorem (Hurewicz). *Suppose X is a simplicial space such that*

$$\pi_0(X) = \pi_1(X) = \ldots \pi_{n-1} = 0, \quad n \geq 2.$$

Then there exists an isomorphism $h : \pi_n(X) \to H_n(X)$.

Proof. To construct h, let x_0 be some point of X (which is path connected because $\pi_0(X) = 0$) and let $\alpha : (S^n, s_0) \to (X, x_0)$ be a spheroid. By the simplicial approximation theorem, we can assume that α is simplicial, and so it induces a homomorphism in simplicial homology $\alpha_* : H_n(\mathbb{S}^n) \to H_n(X)$. But we know that $H_n(\mathbb{S}^n) \cong \mathbb{Z}$, where $1 \in \mathbb{Z}$ corresponds to the identity map of the n-sphere. Now we can define h by putting $h(\alpha) := \alpha_*(\mathrm{id})$.

The fact that h is well defined is obvious, while the proof of its injectivity and surjectivity is a mildly difficult problem for the reader (see the exercise class). The map h is also known as the *Hurewicz isomorphism.*

6.8. Problems

6.1. Prove the Splitting Lemma.

6.2. Prove the Five-Lemma.

6.3. Prove the strong form of the Five-Lemma, in which instead of the commutativity of the diagram we require its commutativity up to sign, e.g. $q \circ f = \pm f' \circ p$, etc.

6.4. Prove the 3-by-3 Lemma.

6.5. Prove the Short-to-Long Exact Lemma.

6.6. Suppose $f : (X, A) \to (Y, B)$ is a simplicial map such that the maps $f|X$ and $f|A$ are homotopy equivalences. Prove that $H_n(X, A) \cong H_n(Y, B)$.

6.7. Give an example of two simplicial maps for which there is no acyclic carrier. Give a geometric explanation of the fact that there is no acyclic carrier in your example.

6.8. Compute the homology of the n-sphere by using the Mayer–Vietoris sequence.

6.9. Compute the homology of the 2-torus $\mathbb{T}^2$ by using the homotopy invariance of homology and the Mayer–Vietoris sequence.

6.10. Compute the 1-homology of the wedge of two circles knowing that $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{Z} * \mathbb{Z}$ (here $*$ stands for the free product).

6.11. Give an example of a Hurewicz homomorphism in dimension $n \geq 2$ which is not injective.

6.12. Give an example of a Hurewicz homomorphism in dimension $n \geq 2$ which is not surjective.

Lecture 7

SINGULAR HOMOLOGY

In this lecture, we develop singular homology, which is a very general theory (it deals with arbitrary topological spaces and continuous maps) and is as simple to construct as simplicial homology. (For this reason, many books on homology theory *start* with an exposition of singular homology.) To my mind, the main defect of that approach is that within its framework, at first, meaningful examples of computations based on the basic definitions are practically impossible; also, it hides the geometric meaning of homology.

We will learn, however, that singular homology satisfies a series of conditions (known as the Steenrod-Eilenberg axioms) which actually uniquely determine homology theories. The corresponding uniqueness theorem implies that the singular theory yields the same homology groups and the same induced homomorphisms as other homology theories, in particular simplicial and cellular homology, so that the results of computations can be borrowed from the previous lectures and exercises.

7.1 Main definitions and constructions

Let X be an arbitrary topological space. Let $\Delta^n = [0, 1, \ldots, n]$ be the standard n-simplex, i.e., the convex hull of the set consisting of the origin 0 and the extremities (denoted by $1, \ldots, n$) of the basis vectors of $\mathbb{R}^n$. We denote the face of Δ opposite to the ith vertex by

$$\Delta\langle i\rangle := \big[0, \ldots, (i-1), (i+1), \ldots, n\big], \quad i = 0, \ldots, n.$$

A *singular n-simplex* σ is any continuous map $\sigma : \Delta^n \to X$. The set of all singular n-simplices is denoted by Σ^n. Let G be a commutative ring with unit. By a *singular n-chain* we mean any finite formal linear combination of singular n-simplices with coefficients from $\mathbb{G}$, and write

$$C_n(X) = \Big\{c = \sum_{\sigma_\alpha \in \Sigma^n} g_\alpha \sigma_\alpha, \mid g_\alpha \in \mathbb{G}\Big\}.$$

The set $C_n(X)$ has a natural $\mathbb{G}$-module structure, with the sum of two chains $c = \sum g_\alpha \sigma_\alpha$ and $c' = \sum g'_\alpha \sigma_\alpha$ defined by

$$c + c' := \sum (g_\alpha + g'_\alpha)\sigma_\alpha$$

and multiplication by a constant $\lambda \in \mathbb{G}$ by $\lambda c = \sum(\lambda g_\alpha)\sigma_\alpha$.

Next we define the *boundary operator* $\partial_n : C_n(X) \to C_{n-1}(X)$ on each simplex σ by setting

$$\partial_n(\sigma) = \sum_{k=0}^{n} (-1)^k \, \sigma\langle k\rangle,$$

where $\sigma\langle k\rangle := \sigma|_{\Delta\langle k\rangle}$ denotes the restriction of σ to the kth face of Δ (which is of course a singular $(n-1)$-simplex); then we extend ∂_n to the whole group $C_n(X)$ by linearity.

Doing this for all $n \geq 0$, we obtain a sequence of Abelian groups (in fact, $\mathbb{G}$-modules) and homomorphisms

$$\dots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_1} C_0(X),$$

called the *complex of singular chains* of X and denoted $\mathcal{C} = \big\{(C_n, \partial_n)\big\}$.

Lemma (Poincaré) *The boundary operator of singular chains satisfies*

$$\boxed{\partial_{n-1} \circ \partial_n = 0 \ \text{ for all } n \geq 2,}$$

so that the complex of singular chains is a chain complex.

Proof. By linearity, it suffices to prove that $\partial_{n-1}(\partial_n(\sigma)) = 0$ for any simplex σ. We have

$$\partial_{n-1}\big(\partial_n(\sigma)\big) = \partial_{n-1}\Big(\sum_{k=0}^{n} (-1)^k \, \sigma\langle k\rangle\Big) = \sum_{k,l} (-1)^k \partial_{n-1}(\sigma\langle k\rangle) = 0,$$

because each singular $(n-2)$-simplex $\sigma\langle k, l\rangle$ (obtained from σ by restricting to the $(n-2)$ face of Δ^n which does not contain the kth and the lth vertices) in the last sum occurs twice with opposite signs. □

From the given topological space X, we have obtained a chain complex $\mathcal{C}(X) = \big(C_n(X), \partial_n\big)$, so now we can define the nth homology group of X as the nth homology group of this chain complex (see Lecture 5, Sect. 5.2), i.e., by setting (for any $n \geq 0$)

$$\boxed{H_n(X; \mathbb{G}) := H_n(\mathcal{C}(X)) = (\operatorname{Ker} \partial_n)\big/(\operatorname{Im} \partial_{n+1})}$$

Now suppose we are given a continuous map $f : X \to Y$ of topological spaces. Our goal is to construct homomorphisms f_* (induced by f) of the corresponding homology groups. We begin by doing it on the chain level:

$$f_{*n} : C_n(X) \to C_n(Y), \quad f_{*n}(c) = f_{*n}\Big(\sum_k z_k\sigma_k\Big) := \sum_k z_k(f \circ \sigma);$$

note that the composition $f \circ \sigma$ is of course a singular n-simplex.

Lemma (Chain Morphism Lemma). *The nth boundary operator commutes with the chain maps induced by f in the sense that*

$$\partial_{Y,n} \circ f_{*n} = f_{*(n-1)} \circ \partial_{X,n}.$$

Proof. The proof of the lemma is a straightforward verification of definitions; we omit it. □

The above lemma means that we have constructed a functor from the category of topological spaces and their continuous maps to the category of chain complexes and their morphisms. Now we can define not only the singular homology groups of topological spaces but also their induced homomorphisms via the homology of chain complexes (Lect.5, Sect.5.2). The functoriality of the obtained assignment $\mathcal{T}\mathrm{op} \rightsquigarrow \mathcal{G}\mathcal{G}\mathrm{r}$ means that

$$\big(f \circ g\big)_* = f_* \circ g_* \quad \text{and} \quad (\mathrm{id}_X)_{n*} = \mathrm{id}_{H_n(X)}\,.$$

Now let us consider pairs of topological spaces (X, A) and their maps. By a map $(X, A) \to (Y, B)$ of such pairs we mean a continuous map f from X to Y such that $fi(A) \subset B$; we write $f : (X, A) \to (Y, B)$ in that case.

Any pair (X, A) of topological spaces defines the corresponding *relative singular chain complex* $\mathcal{C}(X, A) = \big(C_n(X)/C_n(A), \partial_n\big)$ (here by abuse of notation we write ∂_n for the boundary operator in $\mathcal{C}(X)$, which is well defined on cosets mod $C_n(A)$). The homology of this chain complex is called the *relative singular homology* of the pair (X, A). We write

$$H_*(X, A; \mathbb{G}) = \bigoplus_{n=0}^{\infty} H_n(X, A; \mathbb{G}) := \bigoplus_{n=0}^{\infty} H_n(\mathcal{C}(X, A)).$$

The relative homology group $H_n(X, \varnothing; \mathbb{G})$, which can be identified with $H_n(X; \mathbb{G})$, is sometimes called the *absolute* homology group of X.

Given a map of pairs $f : (X, A) \to (Y, B)$, the corresponding *induced homomorphisms* f_{*n} of relative chains and relative homology groups (denoted f_*) is defined similarly to the "absolute" induced homomorphisms. As above, we obtain a functor from the category of pairs of topological spaces to the category of chain complexes and, therefore, to the category of graded Abelian groups. We omit the obvious details.

7.2 Main properties (the Steenrod–Eilenberg axioms)

We now establish the main properties of singular homology theory. It will turn out that these properties uniquely determine the singular homology functor, so they may be regarded as axioms for homology theories (they are known as the Steenrod-Eilenberg axioms).

But first we summarize what was done in the previous section. To every topological space, every pair of topological spaces, and their continuous maps, we assigned, for each nonnegative integer n, Abelian groups (called homology groups) and group homomorphisms, and to each pair of spaces (X, A), a homomorphism ∂_* of the nth homology group of the pair (X, A) to the $(n-1)$st homology group of A. In the notation introduced above, this correspondence reads as

$$\begin{array}{rcl} X & \mapsto & H_n(X), \quad n = 0, 1, 2, \ldots \\ (X, A) & \mapsto & H_n(X, A), \quad n = 0, 1, 2, \ldots \\ (X, A) & \mapsto & \partial_* : H_n(X, A) \to H_{n-1}(A), \quad n = 1, 2, \ldots \\ f : X \to Y & \mapsto & f_* : H_n(X) \to H_n(Y), \quad n = 0, 1, \ldots \\ f : (X, A) \to (Y, B) & \mapsto & f_* : H_n(X, A) \to H_n(Y, B), \ \ n = 0, 1, \ldots \end{array}$$

The sum over n of the obtained Abelian groups is denoted

$$H_*(X) := \bigoplus_{n=0}^{\infty} H_n(X) \ \text{ and } \ H_*(X, A) := \bigoplus_{n=0}^{\infty} H_n(X, A);$$

these objects will be called *graded homology groups* of X and (X, A).

Theorem. *The correspondences described above define a covariant functor (called singular homology) from the category $\mathcal{T}op$ of topological spaces to that of graded Abelian groups, and this functor has the following properties.*

(I) *Dimension:* $H_0(pt) = \mathbb{G}$, *where pt is the singleton and* $\mathbb{G}$ *is an Abelian group, and* $H_n(pt) = 0$ *for* $n > 0$.

(II) *Commutation: the following square diagram*

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow[f_*]{} & H_n(Y,B) \\ {\scriptstyle \partial_*}\downarrow & & {\scriptstyle \partial_*}\downarrow \\ H_{n-1}(A) & \xrightarrow[(f|_A)_*]{} & H_{n-1}(B). \end{array}$$

is commutative.

(III) *Homotopy invariance: if two maps $f, g : (X, A) \to (Y, B)$ are homotopic, then $f_* = g_*$, the induced homomorphisms coincide in all dimensions, and so homotopy equivalent spaces have the same homology.*

(IV) *Exactness: For any pair of spaces (X,A), the following sequence is exact:*

$$\dots \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X)$$

$$\xrightarrow{j_*} H_{n-1}(X,A) \xrightarrow{\partial_*} \dots \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X,A).$$

(V) *Excision: Suppose that $U \subset X$ is an open subset whose closure lies in the interior of A, where $A \subset X$. Then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism in homology $H_n(X \setminus U, A \setminus U) \to H_n(X, A)$ in all dimensions n.*

Proof. Property (I) is obvious.

Property (II) follows by a straightforward verification of definitions.

Let us prove (III) in the particular case $A = B = \varnothing$ (the general case is quite similar). Given homotopic maps $f, g : X \to Y$, we shall construct a chain homotopy

$$D_k = C_k(X) \to C_{k+1}(Y);$$

this will imply $f_* \equiv g_*$ by the Chain Homotopy Lemma. Let $H : X \times \mathbb{I} \to Y$ be a homotopy between f and g. To construct D_k, consider a singular simplex $\sigma : \Delta^k \to X$ and the Cartesian product $\Delta^k \times \mathbb{I} =: T$. The set T has a canonical triangulation consisting of $(k+1)$-simplices all of whose vertices lie in $\Delta^k \times \{0\}$ and $\Delta^k \times \{1\}$. For $k = 1$ and $k = 2$, the simplices are shown in Fig.1 below.

For an arbitrary k, we denote by $0, 1, \dots k$ the vertices of Δ^k, by $0_0, 1_0, \dots k_0$ and $0^1, 1^1, \dots k^1$ those of $\Delta^k \times \{0\}$ and $\Delta^k \times \{1\}$, respectively. Then a generic $(k+1)$-simplex of T is of the form $[0_0, \dots j_0, j^1, \dots k^1]$; note that the number

j of the last vertex lying in $\Delta^k \times \{0\}$ is the same as that of the first vertex lying in $\Delta^k \times \{1\}$.

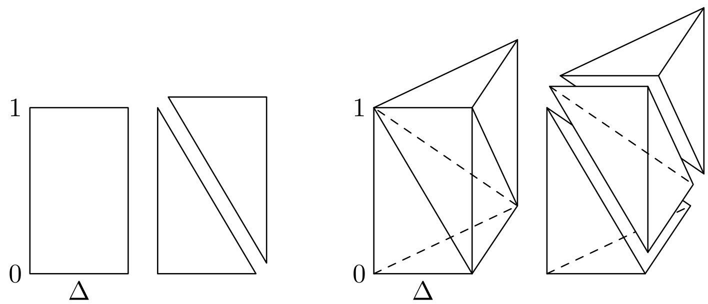


FIGURE 7.1. Triangulations of $\Delta^k \times \mathbb{I}$

Now we consider the homotopy $S : \Delta^k \times \mathbb{I} \to Y$, $S(x,t) := H(f(x),t)$ and define D_k on the singular simplex σ by setting

$$D_k(\sigma) := \sum_{j=0}^{k} (-1)^j S(\delta_j^{k+1}),$$

where δ_j^{k+1} is the linear map from the standard simplex

$$\Delta^{k+1} = [0, 1, \ldots, k+1]$$

that takes j and $(j+1)$ to $\tau(j)_0$ and $\tau(j)^1$, respectively. Thus the right-hand side of the previous displayed formula is a chain in $C_{k+1}(Y)$. Having constructed D_k on an arbitrary singular k-simplex of X, we extend it by linearity to chains from $C_k(X)$.

The fact that this construction indeed yields a chain homotopy, i.e., that

$$\partial_{k+1} D_k + D_{k+1} \partial_k = g_* - f_*,$$

can be established by a tedious but straightforward verification.

Finally, the isomorphism of the corresponding homology groups of homotopy equivalent spaces follows from what was just proved and from the functoriality of our construction.

To prove exactness (IV), let us consider the short sequence of chain complexes

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{i_*} \mathcal{C}(X) \xrightarrow{p_*} \mathcal{C}(X, A) \longrightarrow 0,$$

where i_* is the inclusion homomorphism and p_* is the homomorphism obtained by killing (i.e., supplying with zero coefficients) all singular simplices entirely contained in A. This sequence is obviously exact. Using the Short-to-Long Exact Lemma, we obtain the required long homology sequence,

The proof of (V) (excision) is rather technical (it is based on iterated baricentric subdivisions of standard simplices and a kind of chain homotopy construction); see V.Prasolov, *Elements of Homology Theory*, pp.198-200. We omit it. □

7.3 Uniqueness Theorems

There are many (co)homology theories based on completely different approaches. Besides cellular, simplicial, and singular, there is Vietoris homology (for metric spaces), Čech homology (for topological spaces), Dowker homology (for arbitrary relations), de Rham cohomology (for smooth manifolds), etc. To give an answer to the natural question: (i.e., do these different approaches yield the same groups and homomorphisms?) one would like to have an axiomatic characterization of homology theory functors.

Such a characterization was obtained by N.Steenrod and S.Eilenberg in the 1960ies and consists of five axioms. These axioms coincide with the five properties of singular homology theory appearing in the theorem proved in the previous section and are known as the *Steenrod-Eilenberg axioms.* Using them, one can state and prove different uniqueness theorems, such as

- *On the category of topological spaces and maps, a functor satisfying axioms* (I) – (V) *is unique in the sense that it produces the same groups* $H_*(X, A)$ *and the same induced homomorphisms* f_* *as the singular homology functor constructed in the previous section.*

- *On the category of simplicial spaces and maps, a functor satisfying axioms* (I) – (V) *is unique in the sense that it produces the same groups* $H_*(X, A)$ *and the same induced homomorphisms* f_* *as the simplicial homology functor described in the previous lecture.*

We shall not prove these (or any other) uniqueness theorems, but let us briefly sketch the idea of the proof of the second one for finite simplicial spaces (i.e., simplicial spaces consisting of a finite number of simplices). Using the dimension axiom, we can write $H_n(\text{pt}) = \mathbb{G}$. Since the n-simplex can be contracted to a point, homotopy invariance implies that its homology is the same as that of the point. Using induction and the Mayer–Vietoris sequence, it is not difficult to find the homology of the n-sphere. Given a finite simplicial space X, we inductively construct it by pasting together simplices of different dimensions one after another, computing the homology along the way; in the process, we repeatedly use the Mayer-Vietoris sequence (see below).

Remark 1. It immediately follows from the above uniqueness theorem that simplicial homology is homotopy invariant in the topological sense, i.e., any homotopic continuous maps induce the same homomorphism in homology. (Recall that in the previous lecture we proved, for simplicial homology, a weaker form of homotopy invariance; namely, we established invariance only for PL-homotopic maps.) Now we can assert in particular that the simplicial homology groups do not depend on the triangulation of the given simplicial space and homeomorphic simplicial spaces have the same homology.

Remark 2. The first of the Steenrod–Eilenberg axioms (which asserts that the n-homology of the point is zero for $n > 0$ and seems rather trivial) is actually of great importance: by replacing it by different other statements one obtains such important theories as *extraordinary homology*, *K-theory*, and (*co*)*bordism* theory (which are outside the scope of this course).

6.6. Mayer–Vietoris sequence in singular homology

In singular homology, we also have a Mayer–Vietoris sequence, but its formulation is more delicate (see Problem 7.3) and requires an technical condition concerning the subsets $X_1, X_2 \subset X$, $X_1 \cup X_2 = X$. We say that such a pair satisfies the *excision condition* if the natural chain map of the chain group $\mathcal{C}_*(X_1) + \mathcal{C}_*(X_2)$ (consisting of sums of chains from X_1 and X_2) to $\mathcal{C}_*(X_1 \cup X_2)$ induces an isomorphism in homology.

Theorem (Mayer–Vietoris). *Let X_1 and X_2 be subspaces of the topological space X whose union is X and which satisfy the excision condition for pairs. Then we have the following exact sequence:*

$$\dots \xrightarrow{\partial_*} H_n(X_1 \cap X_2) \xrightarrow{i_*} H_n(X_1) \oplus H_n(X_2) \xrightarrow{j_*} H_n(X) \xrightarrow{\partial_*} \dots,$$

Proof. The argument is similar to the proof in the simplicial case (it is based on the Short-to-Long Exact Lemma), except that the third term of the short exact sequence is $\mathcal{C}_*(X_1) + \mathcal{C}_*(X_2)$ rather than $\mathcal{C}(X_1 \cup X_2)$. $\square$

7.4. Problems

7.1. Prove the Chain Morphism Lemma.

7.2. Give an example showing that the following version of the excision property does not hold: suppose that $U \subset X$ is an open subset that lies in A, where $A \subset X$; then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism in homology $H_n(X \setminus U, A \setminus U) \to H_n(X, A)$ in all dimensions n.

7.3. Show that the Mayer–Vietoris sequence in singular homology does not hold if we only assume that $Y = X_1 \cap X_2$.

7.4. Prove that $\widetilde{H}_k(\mathbb{S}^n; \mathbb{Z}) = 0$ for $k < n$ and is isomorphic to $\mathbb{Z}$ for $k = n$; here $\widetilde{H}_k$ is the reduced singular homology group.

7.5. Prove that $H_k(\mathbb{D}^n, \partial\mathbb{D}^n; \mathbb{Z}) = 0$ if $k \neq n$ and is isomorphic to $\mathbb{Z}$ otherwise.

7.6. Construct and justify the *suspension isomorphism* $H_k(\Sigma) \cong \widetilde{H}_{k-1}(X)$.

7.7. Prove that $H_k(X, A) \cong H_k(X \cup CA, CA)$, where CX (CA) denotes the cone over X (over A).

7.8. Prove that $H_k(X, A) \cong H_k(X \cup CA)$ for $k > 0$; here CA denotes the cone over A.

7.9. Suppose that X is a connected CW-space and A is its CW-subspace. Prove that $H_k(X.A) \cong \widetilde{H}_k(X/Y)$.

7.10*. Suppose $A \subset \mathbb{R}^n$ is closed and does not coincide with $\mathbb{R}^n$; then $\widetilde{H}_{k+i}(\mathbb{R}^{n+i} \setminus A) \cong \widetilde{H}_k(\mathbb{R}^n \setminus A)$; here $\mathbb{R}^n$ is naturally embedded in $\mathbb{R}^{n+i}$.

7.11. Prove the following theorem (due to Alexander): if A and B are homeomorphic closed sets in $\mathbb{R}^n$, then $H_k(\mathbb{R}^n \setminus A) \cong H_k(\mathbb{R}^n \setminus B)$.

7.12. Prove the Jordan–Brouwer theorem: if $A \subset \mathbb{R}^n$ is homeomorphic to the $(n-1)$-sphere, then $\mathbb{R}^n \setminus A$ consists of two connected components.

7.13. Show that the excision axiom does not hold for homotopy groups.

Lecture 8

APPLICATIONS OF HOMOLOGY

In this chapter, we study some simple applications of homology theory, mostly to simplicial spaces, in particular smooth manifolds. We begin by giving a geometric interpretation of the homology groups in the lowest (i.e., zero) and highest (i.e., n for n-manifolds) dimensions, namely connectedness and orientability. We learn how to decompose integer homology groups of finite simplicial spaces X into the direct sum of $b \in \mathbb{N}$ copies of $\mathbb{Z}$ (b is the so-called Betti number) and a finite Abelian group (called the torsion group of X). We then investigate the Euler characteristic and learn that it is in fact a deep homological parameter of arbitrary spaces, including those not possessing any triangulation. Finally, we prove the Lefschetz fixed point theorem, one of the most important applications of homology theory.

In this lecture, it will be convenient for us to move from singular to simplicial homology and back depending on the context. By the uniqueness theorem, the results remain valid in either one of the theories (and in fact in the other homology theories as well).

8.1. Connectedness

Homology theory gives a simple characterization of path connectedness for arbitrary topological spaces and indicates the number of path connected components of arbitrary topological spaces.

Theorem. *A topological space X is path connected if and only if the 0-homology group $H_0(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$; this condition can be replaced by $H_0(X;\mathbb{G}) \cong \mathbb{G}$ provided that the coefficient group $\mathbb{G}$ is a ring with unit.*

Proof. To prove necessity, choose a basepoint p in X. Then any other point q can be joined to p by a path, this path may be regarded as a singular simplex Δ^1; then the 1-chain $1 \cdot \Delta^1$ has the boundary $q - p$, so that all points (regarded as 0-chains) are homological to p, and therefore p generates $H_0(X) \cong \mathbb{Z}$. The proof of sufficiency is left for the exercise class. □

Corollary. *The number of path connected components of an arbitrary topological space X is equal to the dimension of the linear space $H_0(X;\mathbb{R})$.*

Proof. Because of the previous theorem, it suffices to prove that no two vertices located in different components are homologous. Assuming the converse, one can easily obtain a path joining the two vertices. □

8.2. Orientability

Let M^n be a triangulated n-dimensional manifold, i.e., a simplicial space each point of which possesses a neighborhood homeomorphic to $\mathbb{R}^n$. We assume known that any smooth manifold has a triangulation and so does any topological manifold of dimension two or three (the last fact is actually a very difficult theorem proved by Edwin Moise in the 1940ies). We say that M^n is *oriented* if a coherent orientation of its n-simplices is given, i.e., all its n-simplices are oriented in such a way that the two orientations induced on any $(n-1)$-simplex by the two adjacent n-simplices are opposite. If such an orientation exists, the manifold is called *orientable*. It is easy to see that if M^n is connected, then there are only two possible choices of orientation on M^n.

The simplest example of a non-orientable manifold is the projective plane $\mathbb{R}P^2$, while the n-sphere $\mathbb{S}^n$ is, of course, orientable.

Theorem. *A connected triangulated n-dimensional manifold M^n is orientable if and only if its n-homology group $H_n(M^n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.*

Proof. To prove the "if part", consider a coherent orientation of all the n-simplices of M^n and take the chain $c_1 \in C_n(M^n;\mathbb{Z})$ with coefficient $+1$ on each of them. This chain is obviously a cycle, and so is any chain obtained by replacing the plus ones by a fixed integer $z \in \mathbb{Z}$. There are no other cycles, because if the chain has non-constant coefficients, then (using the fact that M_n is connected) we can find two adjacent n-simplices with different coefficients; in that case the boundary of the cycle will be nonzero on their common face. The cycle c_1 clearly generates the group $H_n(M^n;\mathbb{Z}) \cong \mathbb{Z}$.

The "only if" part will be proved in the exercise class. □

For an orientable smooth n-manifold M, the choice of the generator of $H_n(M) \cong \mathbb{Z}$ defines one of the two orientations on M; this generator is called the *fundamental class* of M. If M and N are both oriented and $f : M \to N$ is a map, then the image under f_* of the fundamental class $\varphi \in H_n(M)$ is of the form $d \cdot \psi$, where $\psi \in H_n(N)$ is the fundamental class of N and d is an integer; this integer is called the *degree* of f and denoted $\deg(f)$. The fact that in the case $M = N = \mathbb{S}^n$ this definition agrees with the geometric one for neat maps given in Lecture 4 will be proved in the exercise class.

8.3. Betti numbers and torsion subgroup

In what follows, we shall need a classical algebraic fact about Abelian groups, which we now state without proof.

Algebraic fact. *Any finitely generated Abelian group G can be expressed in the form*

$$\mathbb{G} = \bigoplus_{i=1}^{b} \mathbb{Z}_{(i)} \oplus T,$$

where $b \in \mathbb{N}$ is a nonnegative integer, called the rank of $\mathbb{G}$, the $\mathbb{Z}_{(i)}$ are copies of $\mathbb{Z}$, and T is a finite Abelian group called the torsion of $\mathbb{G}$.

The above algebraic fact immediately implies the following theorem.

Theorem. *If X is a finite simplicial space, then its integer homology can be expressed as follows:*

$$H_k(X;\mathbb{Z}) = \bigoplus_{i=1}^{b} \mathbb{Z}_{(i)} \oplus T,$$

where $b \in \mathbb{N}$ is a nonnegative integer, called the kth Betti number of X, the $\mathbb{Z}_{(i)}$ are copies of $\mathbb{Z}$, and T is a finite Abelian group called the kth torsion group of X.

Proof. The chain group $C_k(X;\mathbb{Z})$ is finitely generated (by the k-simplices of X) and therefore so are the group of cycles and boundaries in dimension k. Hence $H_k(X;\mathbb{Z})$ is finitely generated (and Abelian), so the assertion follows from the previous algebraic fact. □

8.1. Euler characteristic

The *Euler characteristic* of a finite n-dimensional simplicial space is the alternated sum of the number k_i of its simplices in each dimension i, i.e.,

$$\chi(X) := k_0 - k_1 + k_2 - \cdots + (-1)^n k_n.$$

This integer invariant is the oldest one in topology, it was actually first invented and computed by Descartes (and only a century later by Euler) in the case of convex polyhedra. Apparently Riemann was the first to generalize it to other simplicial spaces, in particular to two-dimensional manifolds (Riemann surfaces). The reader should know that $\chi(M)$ classifies orientable 2-manifolds M.

Consider the sequence

$$0 \longrightarrow Z_k(X;\mathbb{F}) \xrightarrow{\ i\ } C_k(X;\mathbb{F}) \xrightarrow{\ \partial\ } B_{k-1}(X;\mathbb{F}) \longrightarrow 0,$$

where $\mathbb{F}$ is a field and C_k, Z_k, B_{k-1} are the chain, cycle, and boundary groups, respectively; its exactness implies that $\dim Z_k + \dim B_{k-1} = \dim C_k$. On the other hand, we obviously have $\dim C_k = a_k$. Therefore

$$\chi(X) = \sum(-1)^k C_k = \sum(-1)^k Z_k + \sum(-1)^k B_{k-1} = \\ = \sum(-1)^k\big(\dim Z_k - \dim B_{k-1}\big) = \sum(-1)^k \dim H_k(X;\mathbb{F}).$$

because $\dim H_k(X;\mathbb{F}) = Z_k/B_{k-1}$.

The case of integer homology is similar and is left as an exercise. □

Corollary. *The Euler characteristic is homotopy* (*and therefore topologically*) *invariant. In particular, it does not depend on the triangulation of the given space.*

We can now *define* the Euler characteristic of *any* topological space X with finitely generated homology $H_*(X;\mathbb{R})$ over the field $\mathbb{R}$, e.g. of any smooth compact n-manifold, as the alternated sum of its Betti numbers (see the displayed formula above), thus obtaining a simple but very deep integer invariant of that space.

8.4. Lefschetz fixed point theorem

Suppose X is a finite simplicial space and $f : X \to X$ is a continuous map. Then the induced map $f_* : H_k(X;\mathbb{R}) \to H_k(X;\mathbb{R})$ is a linear operator on a finite-dimentional vector space over the field $\mathbb{R}$ and we can consider its trace $\mathrm{tr}(f_{k*})$. We now define the *Lefschetz number* $\Lambda(f)$ by setting

$$\Lambda(f) := \sum_{k\geq 0}(-1)^k \mathrm{tr}(f_*). \tag{1}$$

It follows immediately from the definitions that the Lefschetz number is a generalization of the Euler characteristic, namely the Euler characteristic is the Lefschetz number of the identity map: $\Lambda(\mathrm{id}_X) = \chi(X)$.

The above definition of the Lefschetz number involves the induced homomorphism f_{k*} *in homology*, but we can also consider the homomorphism $f_k : C_k(X;\mathbb{R}) \to C_k(X;\mathbb{R})$ *on the chain level*; f_k is a linear operator on the space $C_k(X;\mathbb{R})$, which is the vector space generated by the (finite) family $\Delta_1, \dots, \Delta_{N_k}$ of k-simplices of X; the matrix $((f_{ij}))$ of this operator in the basis $\{\Delta_i\}$ has a simple geometric meaning (when f is a simplicial map): f_{ij} equals ± 1 if Δ_i is mapped onto Δ_j with the same (opposite) orientation and

f_{ij} equals 0 if Δ_i is not mapped onto Δ_j. Now for a simplicial map $f : X \to X$ we can replace f_* by f_k in definition (1), but it turns out (surprisingly!) that the result will be the same, as the following lemma asserts.

Lemma [Hopf]. *Suppose X is a finite simplicial space and a chain map $f_k : C_k(X;\mathbb{R}) \to C_k(X;\mathbb{R})$ is given; then*

$$\sum_{k\geq 0}(-1)^k \mathrm{tr}(f_*) = \sum_{k\geq 0}(-1)^k \mathrm{tr}(f_k). \tag{2}$$

Proof. Consider the chain complex $\{C_k(X;\mathbb{R}), \partial_k\}$; denote $Z_k := \mathrm{Ker}\,\partial_k$ and $B_k := \mathrm{Im}\partial_{k+1}$. For an appropriate subspace $\widehat{C}_k$ of the vector space C_k, we can write $C_k = Z_k \oplus \widehat{C}_k$. The operator f_k maps Z_k to itself and therefore we have a linear operator $\widehat{f}_k : \widehat{C}_k \to \widehat{C}_k$ such that

$$\mathrm{tr} f_k = \mathrm{tr}(f_k|_{Z_k}) + \mathrm{tr}\widehat{f}_k. \tag{3}$$

For an appropriate subspace $\widehat{Z}_k$ of the vector space Z_k, we can write $Z_k = B_k \oplus \widehat{Z}_k$. Then obviously $\widehat{Z}_k \cong H_k(X;\mathbb{R})$, and the operator induced by f_k on $H_k(X;\mathbb{R})$ coincides with f_*. Therefore we have

$$\mathrm{tr}(f_k|_{Z_k}) = \mathrm{tr}(f_k|_{B_k}) + \mathrm{tr}\widehat{f}_*. \tag{4}$$

Now the fact that $\partial_k : C_k \to B_{k-1}$ is a boundary operator of a chain complex shows that we have an isomorphism $\widehat{C}_k \to B_{k-1}$ and that $\widehat{f}_k$ actually coincides with f_{k-1}. Combining this with equations (3) and (4), we obtain

$$\mathrm{tr}(f_k) = \mathrm{tr}(f_k|_{B_k}) + \mathrm{tr}(f_*) + \mathrm{tr}(f_{k-1}|_{B_{k-1}}).$$

Summing this over k with alternating signs, we obtain (1), because the first and third sum in the right-hand side cancel each other out.

Theorem (Lefschetz). *A continuous map $f : X \to X$ of a finite simplicial space to itself with nonzero Lefschetz number $\Lambda(f) \neq 0$ has a fixed point.*

Proof. Suppose that the map f has no fixed points. By compactness, there is a positive lower bound for the distance between points and their images. Therefore, if we take the iterated baricentric subdivision of X a sufficient number of times (denoting by X' the obtained simplicial space) and consider the simplicial approximation $\varphi : X' \to X'$ of f, we can assume that $\Delta' \cap \varphi(\Delta') = \varnothing$ for any simplex $\Delta' \in X'$. But then the diagonal of the matrix of the linear operator $\varphi_k : C_k(X;\mathbb{R}) \to C_k(X;\mathbb{R})$ consists of zeros, so that $\mathrm{tr}(\varphi_k) = 0$ for all k. The theorem follows by the Hopf Lemma. □

Corollary. *Any continuous map of an acyclic finite simplicial space has a fixed point.*

Note that the Brouwer fixed point theorem is a simple particular case of this corollary.

8.6. Vector fields on spheres

In the Topology-1 course, we proved that there are no continuous vector fields without singular points on the two-dimensional sphere $\mathbb{S}^2$. That theorem has the following generalization.

Theorem. *There are no continuous vector fields on the even-dimensional sphere* $\mathbb{S}^{2k}$, $k \geq 1$.

Proof. Suppose that such a field $v(x)$ on $\mathbb{S}^{2k}$ exists. Regard the sphere $\mathbb{S}^{2k}$ as the standard sphere in $\mathbb{R}^{2k+1}$ and define the map $f : \mathbb{S}^{2k} \to \mathbb{S}^{2k}$ by assigning to each point $x \in \mathbb{S}^{2k}$ the intersection with $\mathbb{S}^{2k}$ of the ray issuing from the origin and passing through the end point of the vector $v(x)$.

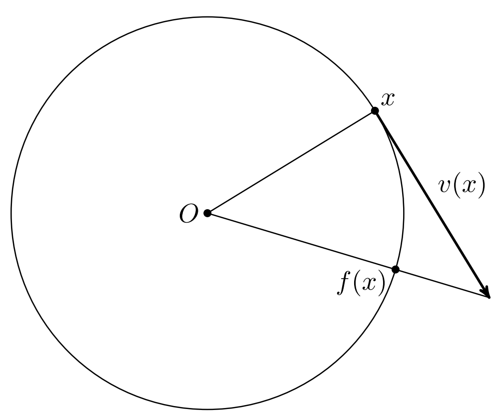


FIGURE 8.11. Construction of the map f

The map f is obviously continuous and has no fixed points, is homotopic to the identity, and therefore $\deg(f)$=1. However, by the definition of the Lefschetz number, we have

$$\Lambda(f) = 1 + (-1)^n \deg(f)$$

for any map $f : \mathbb{S}^n \to \mathbb{S}^n$; so in our case ($n = 2k$) we obtain $\Lambda(f) = 2 \neq 0$. By the Lefschetz Theorem, this is a contradiction. □

The question of the existence of nonsingular continuous vector fields on *odd-dimensional* spheres will be discussed in the exercise class.

8.5. Problems

8.1. Prove that a topological space with zero-dimensional homology isomorphic to $\mathbb{Z}$ is path connected.

8.2. Prove that a connected triangulated n-manifold with n-homology group isomorphic to $\mathbb{Z}$ is orientable.

8.3. Prove that the rank of the $\mathbb{Z}$-module $H_k(X;\mathbb{Z})$ is equal to the dimension of the vector space $H_k(X;\mathbb{R})$.

8.4. Prove that the degree of sphere maps as defined geometrically in Lecture 4 for neat spaces coincides with the homological definition of degree given in the present lecture.

8.5. Compute the Euler characteristic of the n-sphere.

8.6. Using the Mayer–Vietoris sequence, prove that

$$\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_1 \cap X_2),$$

where X_1 and X_2 are simplicial subspaces of a simplicial space X. Prove the same equality "by counting simplices".

8.7. Compute $\chi(\mathbb{C}P^n)$.

8.8. Compute $\chi(\mathbb{R}P^n)$.

8.9. Prove that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ for any finite simplicial complexes X and Y.

8.10. Prove that $\chi(M) = 0$ for any odd-dimensional manifold M.

8.11. Prove that the Euler characteristic of the complement to a knot (or a link) in $\mathbb{S}^3$ is zero.

8.12. Prove that any continuous map of $\mathbb{R}P^n$ to itself has a fixed point if n is even.

8.13. Do there exist continuous maps of $\mathbb{R}P^n$ to itself with no fixed points if n is odd?

8.14. Find a continuous vector field without any singular points on $\mathbb{S}^3$.

8.15. Are there continuous vector fields without any singular points on $\mathbb{S}^{2k+1}$ for $k > 1$?

Lecture 9

COHOMOLOGY

Cohomology groups are dual to homology groups in the same sense that covectors are dual to vectors: they are linear functionals on homology. At first glance, it seems useless to construct a dual theory which is, in a sense, equivalent to the original one (in particular, it satisfies a dual version of the same Steenrod–Eilenberg axioms). However, it turns out that in cohomology theory there is a multiplication operation (the cup product) which has much better properties than the corresponding operation (the cap product) in homology. Moreover, cohomology is the natural setting for other operations (such as Steenrod squares) and for such constructions as the Poincaré isomorphism, and it coincides with the purely analytic cohomology of de Rham defined on smooth manifolds by means of differential forms.

9.1. Definitions and constructions

For the sake of simplicity, we will construct cohomology theory for the case of finite simplicial complexes, although the general case of arbitrary topological spaces can be treated in the same way with a few modifications. (Beware: some modifications are a bit delicate, e.g. the question of dualizing infinite sums of Abelian groups.) You will see in what follows that in practice the dualization of homology theory is formally quite simple, and it consists in *lifting the indices* and *reversing the arrows*.

Let X be a simplicial space and let $\mathbb{G}$ be an Abelian group. We shall write $C_k(X) = C_k(X;\mathbb{Z})$ for the simplicial chain group. A homomorphism $c^k : C_k \to \mathbb{G}$ is called a k-dimensional *cochain* with values in $\mathbb{G}$. Cochains obviously form a group (with the group structure "inherited" from $\mathbb{G}$) denoted by $C^k(X;\mathbb{G})$. In more compact notation, $C^k(X;\mathbb{G}) := \mathrm{Hom}(C_k(X), \mathbb{G})$.

Let $c^k \in C^k(X;\mathbb{G})$ and $c_k \in C_k(X)$. We denote the value of the homomorphism c^k on the chain c_k by $\langle c^k, c_k\rangle$ and define the *coboundary operator*

$$\delta : C^k(X;\mathbb{G}) \to C^{k+1}(X;\mathbb{G}), \quad \langle \delta c^k, c_{k+1}\rangle = \langle c^k, \partial c_{k+1}\rangle.$$

(Here we have omitted the index k in the notation for δ and ∂ because it is clear from the context.) To compute the value of a k-cochain, it suffices (by linearity) to know its value on a k-simplex $[v_0, \dots, v_k]$. This value is given by the formula

$$\delta c([v_0, \dots, v_k]) = \sum_{i=0}^{k+1} (-1)^i \langle c^k, [v_0, \dots, \overset{\vee}{v_i}, \dots v_k]\rangle;$$

the (easy) proof of this formula will be discussed in the exercise class.

The Poincaré Lemma $\partial \circ \partial = 0$ implies $\delta \circ \delta = 0$, and so we can define the *cohomology groups*

$$\boxed{H^k(X;\mathbb{G}) := \frac{Z^k(X;\mathbb{G})}{B^k(X;\mathbb{G})}, \quad H^0(X;\mathbb{G}) := Z^0(X;\mathbb{G})},$$

where Z^k and B^k are the kernel and image of the corresponding coboundary operators in C^k; elements of Z^k are called *cocycles* and those of B^k, *coboundaries*.

We defined cohomology groups on the chain-cochain level, however, for the case of fields (in that case the homology groups are finite-dimensional linear spaces), we could have defined them directly as dual spaces to the homology spaces. This fact is expressed by the following statements, whose simple proofs are omitted.

Theorem (Duality) *If $\mathbb{G}$ is the additive group of a field, then $H^k(X;\mathbb{G})$ is the dual space of $H_k(X;\mathbb{G})$.*

Corollary. *If $\mathbb{G}$ is the additive group of a field and the space $H_k(X;\mathbb{G})$ is finite-dimensional, then $H^k(X;\mathbb{G}) \cong H_k(X;\mathbb{G})$.*

This corollary begs the following question: what is the use of cohomology groups if they are (isomorphic to) the homology groups? The answer to that question was given at the beginning of this lecture – because of the multiplication operation (see Sec.9.4).

The remaining basic steps of the theory, namely the definition and/or construction of *relative cochains*, *relative cohomology groups*, *augmentation*, *reduced cohomology*, *induced homomorphisms* in cohomology for spaces and pairs of spaces, the *coboundary operator* $\delta^* : H^k(X, A;\mathbb{G}) \to H^k(X;\mathbb{G})$ is similar (or rather dual) to the corresponding definitions and/or constructions in homology theory and are left as exercises for the reader.

9.2. Properties (Steenrod–Eilenberg axioms for cohomology)

We will now list the main properties of the cohomology theory constructed above (which can also be regarded as the Steenrod–Eilenberg axioms for cohomology) for finite simplicial spaces.

But first we summarize what was done in the previous section. To every finite simplicial space, to every pair of such spaces, and to their simplicial maps, we assigned, for each nonnegative integer n, Abelian groups (called cohomology groups) and group homomorphisms, and to each pair of spaces (X, A), a homomorphism ∂_* of the nth cohomology group of A to the $(n+1)$st cohomology group of the pair (X, A). In the notation introduced above, this correspondence reads as

$$\begin{array}{rcl} X & \mapsto & H^n(X), \quad n = 0, 1, 2, \dots \\ (X, A) & \mapsto & H^n(X, A), \quad n = 0, 1, 2, \dots \\ (X, A) & \mapsto & \partial^* : H^n(X, A) \to H^{n-1}(A), \quad n = 1, 2, \dots \\ f : X \to Y & \mapsto & f^* : H^n(Y) \to H^n(X), \quad n = 0, 1, \dots \\ f : (X, A) \to (Y, B) & \mapsto & f^* : H^n(Y, B) \to H^n(X, A), \ \ n = 0, 1, \dots \end{array}$$

The sum over n of the obtained Abelian groups is denoted

$$H^*(X) := \bigoplus_{n=0}^{\infty} H^n(X) \ \text{ and } \ H^*(X, A) := \bigoplus_{n=0}^{\infty} H^n(X, A);$$

these objects will be called *graded cohomology groups* of X and (X, A).

Theorem. *The correspondences described above define a contravariant functor* (*called the cohomology functor*) *from the category of finite simplicial spaces to that of graded Abelian groups. The contravariance of the functor means that*

$$(f \circ g)^* = g^* \circ f^* \quad \text{and} \quad (id_X)^* = id_{H^n(X)}.$$

This functor has the following properties.

(I) *Dimension:* $H_0(pt) = \mathbb{G}$, *where* pt *is the singleton and* $\mathbb{G}$ *is an Abelian group, and* $H_n(pt) = 0$ *for* $n > 0$.

(II) *Commutation: the following square diagram*

$$\begin{array}{ccc} H^n(X, A) & \xrightarrow[f^*]{} & H^n(Y, B) \\ {\scriptstyle \partial_*}\uparrow & & {\scriptstyle \partial_*}\uparrow \\ H^{n-1}(A) & \xrightarrow[(f|_A)^*]{} & H^{n-1}(B). \end{array}$$

is commutative.

(III) *Homotopy invariance: if two maps* $f, g : (X, A) \to (Y, B)$ *are homotopic, then* $f^* = g^*$, *the induced homomorphisms coincide in all dimensions, and so homotopy equivalent spaces have the same cohomology.*

(IV) *Exactness: For any pair of spaces (X,A), the following sequence is exact:*

$$\dots \xleftarrow{i_*} H^n(X) \xleftarrow{j_*} H^n(X,A) \xleftarrow{\delta^*} H^{n-1}(A) \xleftarrow{i_*} H^{n-1}(X)$$
$$\xleftarrow{j_*} H^{n-1}(X,A) \xleftarrow{\delta^*} \dots \xleftarrow{i_*} H^0(X) \xleftarrow{j_*} H^0(X,A).$$

(V) *Excision: Suppose that $U \subset X$ is an open subset whose closure lies in the interior of A, where $A \subset X$. Then the inclusion $(X \setminus U, A \setminus U) \hookrightarrow (X,A)$ induces an isomorphism in cohomology*

$$H^n(X,A) \to H^n(X \setminus U, A \setminus U)$$

in all dimensions n.

Proof. The proofs of all five assertions of this theorem follow by duality from the five assertions in the corresponding theorem in homology (see Sec.7.2). For example, to prove (IV) (exactness), we dualize the short exact sequence of chains

$$0 \longrightarrow \mathcal{C}(A) \xrightarrow{i_*} \mathcal{C}(X) \xrightarrow{p_*} \mathcal{C}(X,A) \longrightarrow 0,$$

obtaining the dual sequence of cochains

$$0 \longleftarrow \mathcal{C}(A) \xleftarrow{i^*} \mathcal{C}(X) \xleftarrow{p^*} \mathcal{C}(X,A) \longleftarrow 0,$$

which is also exact (this follows from the Splitting Lemma applied to the chain sequence). The (long) exact sequence for cohomology is then obtained by applying the Short-to-Long Exact Lemma. □

Remark. We have formulated the Steenrod–Eilenberg axioms in the finite simplicial setting; they also hold in the topological setting, and there are uniqueness theorems similar to those in homology theory.

9.3. Ordered homology theory

To construct the multiplication operation, we need a slight modification of simplicial (co)homology theory, in which we order all the vertices of the given simplicial space X and then, instead of considering all $(n+1)!$ ordered simplices determined by each geometric simplex, we consider only one ordered simplex, namely the one whose vertices are listed in ascending order (w.r.t.

the given ordering of the vertices of X). It is not hard to prove that this modification yields the same homology groups and induced homomorphisms (see Exercise 5.11 above).

Let X be a finite simplicial space and let the coefficient group $\mathbb{G}$ be a commutative associative ring with unit (e.g. $\mathbb{Z}$). Recall that we defined the chain complex $\big(C_n(X), \partial_n\big)$ by using *oriented simplices* (which form the basis of each $\mathbb{G}$-module C_n). We will now consider *ordered simplices* $(v_0, \dots v_n)$, where the order of the vertices v_i is fixed but the v_i are not necessarily pairwise different and define the *ordered chain complex* $\big(\widehat{C}_n(X;\mathbb{G}), \partial_n\big)$ by taking $\widehat{C}_n(X)$ to be the linear combination (with coefficients in $\mathbb{G}$) of ordered simplices and using the same boundary operator ∂ as in the usual theory (see 5.4). The rest of the *ordered (co)homology theory* is constructed similarly to the usual simplicial (co)homology theory. Note, however, that the chain complex in the ordered theory is "infinite to the left" (because repetition of vertices forces us to consider simplices of arbitrarily high "dimension"), but this chain complex has no nontrivial homology in dimensions higher than the dimension of X. Moreover, we have the following theorem.

Theorem. *The homology of the two complexes $\widehat{C}_*(X;\mathbb{G})$ and $C_*(X;\mathbb{G})$ is canonically isomorphic.*

We omit the straightforward proof of this theorem.

9.4. Multiplication (cup product)

As above, let X be a finite simplicial space and the coefficient group $\mathbb{G}$ be a commutative associative ring with unit. The *cup product* of two cochains $c^p \in \widehat{C}^p(X;\mathbb{G})$, $c^q \in \widehat{C}^q(X;\mathbb{G})$ is the cochain $c^p \smile c^q \in \widehat{C}^{p+q}(X;\mathbb{G})$ given by

$$\langle c^p \smile c^q, (v_0, \dots, v_{p+q})\rangle = \langle c^p, (v_0, \dots, v_p)\rangle \cdot \langle c^q, (v_p, \dots, v_{p+q})\rangle.$$

The cup product is obviously bilinear and associative, it has a unit defined as the cochain taking the value 1 (where 1 is the unit of the ring $\mathbb{G}$) at all vertices of X. It can be carried over to cohomology classes due to the following beautiful (and easily verified) lemma.

Lemma ("Leibnitz rule"). $\delta(c^p \smile c^q) = (\delta c^p) \smile c^q + (-1)^p c^p \smile \delta(c^q)$.

The proof of this lemma, as well as the fact that it allows to correctly define the cup product on the cohomology level, is a straightforward exercise. It is also easy to prove that

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

Theorem. *The cup product supplies the graded group $H^*(X;\mathbb{G})$ with the structure of a graded $\mathbb{G}$-module; the cup product is skew commutative in the sense that*

$$\alpha \smile \beta = (-1)^{pq}(\beta \smile \alpha).$$

9.5. De Rham Cohomology

In this section, we recall the definition of de Rham cohomology (usually given in advanced courses in differential geometry or on analysis on smooth manifolds) and state without proof the famous de Rham theorem asserting that this cohomology (defined in purely analytic terms) is isomorphic (for compact smooth manifolds) to singular (or simplicial) cohomology.

Let M^n be a smooth compact n-dimensional closed (=without boundary) manifold, let $\Lambda^k M^n$ be the (linear over $\mathbb{R}$) space of differential forms on M^n. In local coordinates, a differential form $\omega^k \in \Lambda^k M^n$ is expressed as

$$\omega^k = \sum_{i_1 \le \cdots \le 1_k} a_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

Recall that for any two differential forms $\omega_1 \in \Lambda^p M^n$ and $\omega_2 \in \Lambda^q M^n$, their exterior product $\omega_1 \wedge \omega_2$ is defined; it is skew commutative in the sense that

$$\omega_1 \wedge \omega_2 = (-1)^{pq}(\omega_2 \wedge \omega_1)$$

and it supplies the graded vector space $\Lambda^* M^n$ with the structure of a graded $\mathbb{R}$-algebra. Any smooth map $f : M^n \to N^m$ induces a linear map

$$f^* : \Lambda^k N^m \to \Lambda^k M^n.$$

The main tool in the construction of de Rham cohomology is the *differential*

$$d : \Lambda^k M^n \to \Lambda^{k+1} M^n.$$

Given a k-form ω^k in local coordinates, we can define its differential by means of the formula

$$d(\varphi(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k}) := \sum_{i=0}^{n} \frac{\partial \varphi(x)}{\partial x_m} dx_m \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and linearity. It is easily verified that $d \circ d = 0$. A k-form ω^k is called *closed* if $d\omega^k = 0$ and *exact* if there exists a $(k+1)$-form λ^{k+1} such that $d\lambda = \omega$.

The equality $d \circ d = 0$ implies that any exact from is closed, and the quotient space of the space of closed k-forms by exact k-forms is, by definition, the *de Rham cohomology* $H^k_{dR}(M^n)$ in dimension k.

The exterior multiplication operation can be carried over from the level of forms to the cohomology level, because here also the Liebnitz rule holds: for any k-form ω_1 and any l-form ω_2,

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2.$$

Each of the $H^k_{dR}(M^n)$ is a linear space over $\mathbb{R}$, their direct sum $H^*_{dR}(M^n)$ supplied with exterior multiplication is a graded $\mathbb{R}$-algebra.

Theorem (de Rham). *For a smooth compact closed manifold* M^n*, the de Rham cohomology* $H^*_{dR}(M^n)$ *is isomorphic as a graded algebra to the singular* (*or simplicial*) *cohomology graded algebra* $H^*(M^n; \mathbb{R})$.

There are no easy proofs of this beautiful theorem (see V.V.Prasolov's book *Elements of Homology Theory*, pp.289-300).

9.5. Problems

9.1. Prove that

$$\delta c([v_0, \ldots, v_k]) = \sum_{i=0}^{k+1} (-1)^i \langle c^k, [v_0, \ldots, v_i^{\vee}, \ldots v_k] \rangle;$$

9.2. Show that the Poincaré Lemma $\partial \circ \partial = 0$ implies $\delta \circ \delta = 0$.

9.3. Prove the Poincaré Lemma $\partial \circ \partial = 0$ for the ordered simplicial homology theory.

9.4. Prove the Liebnitz rule $\delta(c^p \smile c^q) = (\delta c^p) \smile c^q + (-1)^p c^p \smile \delta(c^q)$.

9.5. Prove that $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.

9.6. Given $X = \bigcup_{i=1}^n A_i$, where the A_i are contractible simplicial subspaces of X, prove that the product $\alpha_1 \smile \cdots \smile \alpha_n$ vanishes for any choice of elements $\alpha_i \in H^{k_i}(X)$, where $k_i > 0$. Use this fact to prove that the cup product in the suspension ΣX for any finite simplicial space is trivial in the sense that the product of any two cohomology classes of positive dimension vanish.

9.7. Suppose that the simplicial subspace $A \subset X$ is a retract of the finite simplicial space X, $i : A \hookrightarrow X$ is the inclusion, and $r : X \to A$ is the retraction. Prove that $H_*(X) = \operatorname{Im} i_* \oplus \operatorname{Ker} r_*$ and $H^*(X) = \operatorname{Im} r^* \oplus \operatorname{Ker} i^*$. Show that in the cohomology ring $H^*(X)$ the subset $\operatorname{Ker} i^*$ is an ideal and $\operatorname{Im} r^*$ is a subring.

9.8. Given the cup products in the algebras $H^*(X)$ and $H^*(Y)$, determine the cup product in $H^*(X \wedge Y)$.

9.9. Prove that $\mathbb{S}^n \vee \mathbb{S}^m$ is not a retract of $\mathbb{S}^n \times \mathbb{S}^m$, where $\mathbb{S}^n \vee \mathbb{S}^m$ is understood as $\mathbb{S}^n \times \{x_0\} \cup \{y_0\} \times \mathbb{S}^m$ and $n, m \geq 1$.

9.10. Prove that the standardly embedded space $\mathbb{R}P^m \subset \mathbb{R}P^n$, $n > m$, is not a retract.

Lecture 10
POINCARÉ DUALITY

Poincaré duality is an isomorphism between the homology groups (of different dimensions) of manifolds. It is a rather amazing relationship, since it says that the inner structure of an arbitrary closed compact manifold is, in a sense, symmetric with respect to the middle dimension: roughly speaking, it asserts that the k-dimensional homology of an n-manifold is isomorphic to its $(n-k)$-dimensional (co)homology. This symmetry is based on an extremely simple geometric construction (see Sec.10.1) and uses the notion of cap product, an operation in a sense dual to the cup product studied in the previous lecture.

10.1. The dual cellular decomposition

To a given a compact closed (=without boundary) n-manifold M^n with a fixed triangulation, we will canonically associate (following Poincaré) a cellular decomposition (i.e., we will supply M^n with a CW-space structure). To this end, we begin by taking the baricentric subdivision of the triangulation and, to avoid confusion, refer to a simplex in the subdivision as a *subsimplex*, saving the word "simplex" for simplices in the original triangulation. Now to any vertex (i.e., 0-simplex) σ^0, we associate the n-cell σ_*^n consisting of all n-subsimplices having a vertex at σ^0. More generally, to any k-simplex σ^k we associate the $(n-k)$-cell σ_*^{n-k} consisting of all $(n-k)$-subsimplices having a vertex at the baricenter of σ^k. For $n=2$, see Figure 10.1.

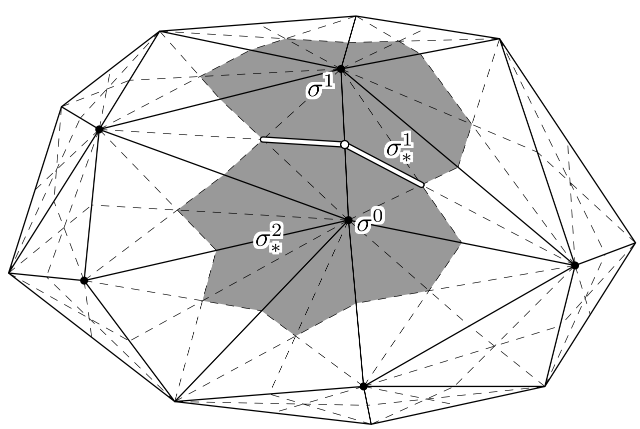


FIGURE 10.1. The dual cellular decomposition for $n=2$

Note that that each simplex σ intersects its dual cell transversally, their only intersection point being the baricenter of σ.

We have not only supplied M^n with a CW-space structure (which we denote by M_*), but we have defined a bijection between the k-simplices σ of the given triangulation and the dual $(n-k)$-cells σ_* of M_*. Moreover, if the orientation of M^n is fixed arbitrarily (i.e., a continuously varying basis, which we will call *positively oriented*, is given at each point), then each oriented simplex σ determines a corresponding orientation of the dual cell in the natural way: we orient the dual cell so that its basis (attached to the baricenter) added to the orienting basis of σ has the same orientation as the positive basis of M^n at the baricenter.

This allows us to construct, for any $k = 0, 1, \dots n$ a homomorphism

$$\delta : C_k(M; \mathbb{Z}) \to C_{n-k}(M_*; \mathbb{Z})$$

given on simplices by $\delta(\sigma^k) = \sigma_*^{n-k}$ and extended by linearity.

The homomorphism δ is defined for all k from 0 to n, so one might naively think that δ is a morphism of the chain complex $\mathcal{C}(M, \mathbb{Z})$ to $\mathcal{C}(M_*, \mathbb{Z})$ which will give us the required isomorphism in homology. But this is completely wrong, because the boundary operators in the chain complexes $\mathcal{C}(M, \mathbb{Z})$ and $\mathcal{C}(M_*, \mathbb{Z})$ act in opposite directions from the point of view of dimension, i.e.,

$$\dim((\partial \circ \delta)(c^k)) = n - k + 1 \neq n - k + 1 = \dim((\delta \circ \partial)(c^k))$$

Having in mind that coboundary operators in cochain complexes also act "in the opposite direction", we will overcome this difficulty by passing from homology to cohomology; this will be done in the next section.

10.2. Homology–cohomology duality

In the rest of this lecture, we will assume that the coefficient group (in homology and cohomology) is the same commutative associative ring R with unit (e.g. $\mathbb{Z}$); the ring R will not explicitly appear in the notation: we write, for instance, $C^k(M_*)$ instead of $C^k(M_*; R)$.

Let us construct an homomorphism φ_k from $C_k(M^n)$ to $C^{n-k}(M_*)$ by assigning to each simplex σ^k (regarded as a k-chain with coefficient $1 \in R$ at σ^k) the linear functional on $(n-k)$-cells that assigns 1 to the cell σ_*^{n-k} dual to σ^k and 0 to the other $(n-k)$-cells. Obviously, φ_k is an isomorphism between $C_k(M^n)$ and $C^{n-k}(M_*)$, and it readily follows from the definitions of φ_k, δ, and ∂ that for all k the diagrams

$$\begin{array}{ccc} C_k(M) & \xrightarrow[\varphi_k]{} & C^{n-k}(M_*) \\ {\scriptstyle \partial_k}\downarrow & & {\scriptstyle \delta_{n-k}}\downarrow \\ C_{k-1}(M) & \xrightarrow[\varphi_{k-1}]{} & C^{n-k+1}(M_*). \end{array}$$

are commutative. This last fact implies in turn that the corresponding homology and cohomology groups are isomorphic. Thus we have proved the following theorem.

Theorem (Poincaré Duality). *Let M^n be a smooth oriented compact closed manifold and R be a commutative associative ring with unit. Then the homology and cohomology groups of complementary dimensions are isomorphic, i.e., for all $k = 0, 1, \ldots, n$,*

$$\boxed{H_k(M; R) \cong H^{n-k}(M; R)}$$

The theorem does not hold for the case in which the manifold M is nonorientable, but if we replace the ring R by $\mathbb{Z}_2$ (the integers mod 2), then we still have an isomorphism, namely

$$H_k(M; \mathbb{Z}_2) \cong H^{n-k}(M; \mathbb{Z}_2) \quad \text{for all} \quad k = 0, 1, \ldots, n.$$

The proof of this fact is an exercise.

The construction of Poincaré duality described above is very simple, but it is defined on the chain-cochain level, it depends on the choice of the triangulation of M, and does not give an effective way to perform computations on the homology-cohomology level. Actually, the Poincaré isomorphism is canonical, and there is a very simple formula for computing it. It requires, however, a new notion (the cap product), which will be introduced in the next section.

10.3. The cap product

The duality isomorphism $\varphi_{k,*} : H_k(M; R) \to H^{n-k}(M; R)$ between the homology and cohomology groups allows to define, for the homology of smooth manifolds, a multiplication operation, called the homology cap product, by simply carrying over to homology the cup product in cohomology via this isomorphism. More precisely, for homology classes $a \in H_p(M)$ and $b \in H_q(M)$, we define their *cap product* $a \frown b \in H_{p+q}(M)$ by setting

$$a \frown b := \varphi_{p+q}^{-1}\big(\varphi_p(a) \smile \varphi_q(b)\big).$$

From this definition and from the properties of the cup product, it immediately follows that *the homology $H_*(M)$ of a smooth compact closed manifold M supplied with the cap product is a graded R-algebra with skew-commutative multiplication.*

This statement is not, as would seem at first glance, a useless dualization. It turns out that the cap product has a simple very visual geometrical interpretation, which sometimes makes its computation simpler than that of the cup product. Also, as we shall see shortly, it gives a simple description of Poincaré's duality isomorphism on the homology-cohomology level.

Now let us give a sketch of the geometric interpretation of the cap product. Suppose $a \in C_p(M)$ and $b \in C_q(M)$ are cycles. To simplify the exposition, we assume that a and b are cycles with coefficients 1 on some simplices and zero on the others; let A be the set of all simplices with coefficient 1 for the cycle a and B the corresponding set for b. Further assume that A and B are *in general position*, which means that two (open) simplices $\sigma^p \in A$ and $\tau^q \in B$ either do not intersect at all or intersect in an open cell of dimension $p+q-n$. Then $A \cap B$ is a cell space. Orient all the cells of $A \cap B$ as follows: first to an orienting basis of $\sigma^p \cap \tau^q$ (it has $p+q-n$ vectors) add $n-p$ vectors to form an orienting basis of σ^p, then add $n-q$ vectors to form an orienting basis of τ^q, in such a way that the n vectors thus obtained determine the chosen orientation of M^n (Fig.10.2).

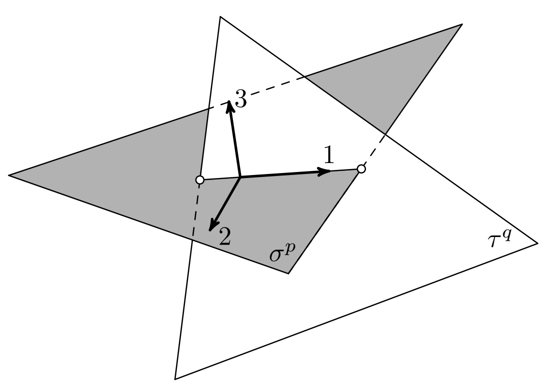


FIGURE 10.2. Orienting the intersection of two simplices

It can be shown that the $(p+q-n)$-cells of $A \cap B$ taken with coefficients 1 form a cycle $a \frown b \in C_{p+q-n}(M^n)$; now if α and β are the homology classes determined by a and b, then the cycle $a \frown b$ determines a well-defined homology class in $H_{p+q-n}(M^n)$ denoted by $\alpha \frown \beta$. Thus we have constructed a bilinear map

$$\frown : H_p(M^n; R) \times H_q(M^n; R) \to H_{p+q-n}(M^n; R).$$

The homology cap product discussed above is more visual but also less general than its other version, the cohomology-homology cap product (usually referred to simply as the cap-product). This operation, also denoted by $\frown$, is defined for the homology and cohomology of path connected simplicial spaces X (not only smooth manifolds) with coefficients in an associative commutative ring R with unit. We will be using *ordered* simplices; to fix an order, we number all the vertices of X once and for all; the vertices of any simplex will always be listed in increasing order. In this setting, the *cap-product* is a bilinear map

$$\frown : H^p(X; R) \times H_{p+q}(X; R) \to H_q(X; R) \tag{1}$$

defined as follows.

Suppose that $c^p \in C^p(X; R)$; as usual, we first define the cap product on simplices; we put

$$c^p \frown [v_0, \ldots, v_{p+q}] := \langle c^p, [v_p, \ldots, v_{p+q}] \rangle [v_0, \ldots, v_q] \in C_q(X; R).$$

In the particular case $q = 0$, we put

$$c^p \frown [v_0, \ldots, v_p] := \langle c^p, [v_0, \ldots, v_p] \rangle \in R = C_0(X, R)$$

Extending this map by linearity, we obtain a bilinear map at the chain-cochain level

$$\frown : C^p(X; R) \times C_{p+q}(X; R) \to C_q(X; R).$$

To obtain a map on the (co)homology level, we need the following version of the Leibnitz rule

Lemma (Leibnitz rule for cap-products).

$$\partial(c^p \frown c_{p+q}) = (-1^q)\big(\delta c^p \frown c_{p+q}\big) + \big(c^p \frown \partial c_{p+q}\big).$$

The proof is a mildly difficult exercise. An immediate consequence of this lemma is the definition of the bilinear map (1), i.e., that of the cap product at the (co)homology level.

The cap product possesses the following properties:

(1) *for any map f of simplicial spaces* $f_*(f^*a \frown b) = a \frown f_*(b)$;

(2) $\langle a^q, b^p \frown c_{p+q}\rangle = \langle a^q \frown b^p, c_{p+q}\rangle$;

(3) $a^p \frown (b^q \frown c_{p+q+r}) = (a^p \smile b^q) \frown c_{p+q+r}$;

here a, b are cohomology classes, c is a homology class.

10.4. The Poincaré isomorphism

Poincaré duality was defined above on the chain-cochain level, and that definition did not give an effective way to perform computations on the (co)homology level. However, the Poincaré isomorphism can be described directly in (co)homology by a simple formula involving the cap product.

Theorem (Poincaré Isomorphism). *The assignment*

$$H^{n-k}(M^n; R) \ni \alpha^{n-k} \mapsto \alpha^{n-k} \frown [M^n] \in H_k(M^n; R),$$

where $[M^n]$ is the fundamental class of the smooth orientable manifold M^n, determines the Poincaré duality isomorphism $H^{n-k}(M^n; R) \cong H_k(M^n; R)$.

Proof. Let K be some triangulation of M^n and K' be its baricentric subdivision. We will be dealing with ordered homology, so we order all the vertices of K' as follows: first we enumerate the vertices of K, then the baricenters of the 1-simplices of K, then the baricenters of the 2-simplices of K, and so on. For the cycle representing the fundamental class of M^n let us take the chain with coefficients 1 on all the n-simplices $(v_0, \dots, v_i)$ of K' written so that the vertices appear in increasing order.

For this ordering, each of the ordered simplices $\sigma^i = (v_0, \dots, v_i)$ of K' is contained in exactly one ordered simplex τ^i of K and the latter is determined by the last vertex v_i of σ^i. The corresponding $(n-i)$-cell τ^i_* (see Sec.10.1) can be represented as the union of simplices $\pm(v_i, \dots, v_n)$ with vertices written in increasing order.

Under Poincaré duality (see Sec.10.1), to each chain $c_k = \sum z_i \sigma_i^k$ we assign the cochain c^{n-k} such that $\langle c^{n-k}, \sigma_{i*}\rangle = z_i$. Therefore,

$$\begin{aligned} c^{n-k} \frown \sum \pm(v_0, \dots, v_n) &= \sum \langle c^{n-k}, \pm(v_k, \dots, v_n)\rangle (v_0, \dots, v_k) \\ &= \langle c^{n-k}, \sigma_{i*}\rangle \sigma_i^k = \sum z_i \sigma_i^k = c_k, \end{aligned}$$

as required. □

10.5. The Gauss linking number

The Gauss linking number is an integer invariant of pairs of smooth curves embedded in $\mathbb{R}^3$ which says how many times one curve "wraps around" the other. Originally defined by Gauss as a double integral in the context of electrodynamics, it has a simple homological interpretation that we now present.

The *Gauss linking number* $\mathrm{lk}(C, D)$ of two disjoint circles C_1 and C_2 embedded in the sphere $\mathbb{S}^3$ is the integer $\mathrm{lk}(C_1, C_2)$ defined as follows: assume (without loss of generality) that C_1 and C_2 are composed of 1-simplices (oriented along the circles) and consider the two chains $c_1, c_2 \in C_1(\mathbb{S}^3; \mathbb{Z})$ with coefficients 1 on these simplices and 0 on the others; obviously, these chains, in particular c_2, are cycles; since $\mathbb{S}^2$ has trivial 1-homology, c_2 is a boundary, i.e., there exists a 2-chain $d \in C_2(\mathbb{S}^2)$ such that $\partial(d) = c_2$; we put

$$\mathrm{lk}(C_1, C_2) := [d \frown c_1] \in H_0(\mathbb{S}^3) = \mathbb{Z}.$$

The fact that the linking number is well defined (does not depend on the choice of d) is the subject matter of an exercise.

Actually $\mathrm{lk}(C_1, C_2)$ is a so-called finite-type invariant in the sense of Vassiliev and can be computed in an elementary way from the diagram of the link $C_1 \cup C_2 \subset \mathbb{S}^3$ by counting the "signs" of the crossing points of the two curves (as shown in Fig.3)

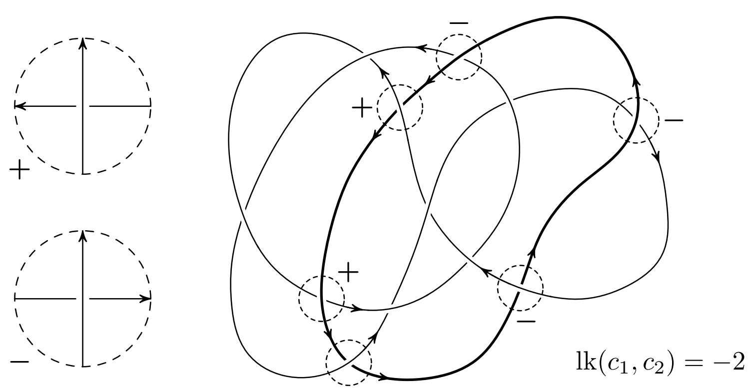


FIGURE 10.3. Counting the Gauss linking number

10.6. Problems

10.1. Let $\varphi : C_k(M) \to C^{n-k}(M_*)$ be the natural homomorphism between the simplicial chain group of a smooth closed manifold M^n and the cochain group of the dual cell decomposition M_* of M^n. Prove the following commutation relation $\delta_{n-k} \circ \varphi_k = \varphi_{k-1} \circ \partial_k$, where ∂ and δ are the boundary and coboundary operators.

10.2. Prove that if $H_1 M^3 = 0$ for a closed 3-manifold M^3, then M^3 is a homology 3-sphere, i.e., its homology groups as $\mathbb{S}^3$.

10.3*. Give an example of a homology 3-sphere which is not homeomorphic to the 3-sphere.

10.4. Let the torus be presented as a CW-space with four cells, let c^1 be the cochain taking one of the 1-cells to 1 and the other to 0, while c_1 is the chain with coefficient 1 at the second 1-cell and 0 at the other. Compute $c^1 \frown c_1$.

10.5. Prove the Leibnitz rule for the cap product.

10.6. Prove the following properties of the cap product:

(i) $f_*(f^*(b) \frown a) = b \frown f_*(a)$, where $f : X \to Y$ is a simplicial map, $a \in H_*X$, and $b \in H^*Y$;

(ii) $\langle a^q, b^p \frown c_{p+q} \rangle = \langle a^q \frown b^p, c_{p+q}$;

(iii) $a^p \frown (b^q \frown c_{p+q+r}) = (a^p \smile b^q) \frown c_{p+q+r}$.

10.7. Give an example of two simplicial complexes with the same homology but with different cap products.

10.8. Prove that the Gauss linking number of two oriented circles embedded in $\mathbb{R}^3$ given by $\operatorname{lk}(C_1, C_2) = \sum \varepsilon_i$, where $\varepsilon = \pm 1$ is as in Fig.3, is well defined.

10.9. Prove the Poincaré duality theorem for nonoriented manifolds.

Lecture 11

OBSTRUCTION THEORY

In the first lecture of this course, we considered the *extension problem*: given a map $f : A \to Y$, where A is a subset of X, to construct a map $F : X \to Y$ that coincides with f on A. If X is a simplicial space (or a CW-space), it is natural to attack this problem by induction: first extend f to the vertices of the triangulation of X, then to the 1-simplices, to the 2-simplices, etc. It turns out that each inductive step can be carried out if and only if a certain cohomology class vanishes. In a sense, this cohomology class "obstructs" our construction, and so the inductive approach described above is called *obstruction theory*.

This theory has numerous applications (not only to the extension of maps), in particular to the classification of homotopy classes of various families of maps by means of the appropriate cohomology groups and to the construction of the so-called Eilenberg–MacLane spaces. The latter, in turn, play a key role on other homotopy classification theorems.

11.1. The obstruction cocycle

Suppose that X is a simplicial space, A is its simplicial subspace, X^n the n-dimensional skeleton of X, and we are given a map $f : \widehat{X}^n \to Y$, where Y is a path connected topological space and $\widehat{X}^n := X^n \cup A$. Our goal is to extend this map to $\widehat{X}^{n+1}$. We will assume that $n \geq 1$, because for a path connected Y the construction of the extension is obvious. Further, we will assume that Y is n-simple, i.e., that the action the $\pi_1(Y)$ on its homotopy groups of dimension n is trivial (this will allow us to add elements of the group $\pi_n(Y, y)$ with different basepoints y).

To the map $f : \widehat{X}^n \to Y$ we assign the cochain

$$c^{n+1}(f) \in C^{n+1}(X; \pi_n(Y))$$

as follows. As usual, it suffices to define c^{n+1} on $(n+1)$-simplices; since each simplex Δ^{n+1} is oriented, its orientation defines an orientation on $\partial(\Delta^{n+1})$ and so the restriction of f to $\partial(\Delta^{n+1})$ can be regarded as a spheroid in Y; the homotopy class of this spheroid is, by definition, the element of $\pi_n(Y)$ that we assign to f. Thus:

$$c^{n+1}(f)(\Delta^{n+1}) := \left[f|_{\partial(\Delta^{n+1})} \right] \in \pi_n(Y).$$

Note that if Δ^{n+1} belongs to A, the spheroid corresponding to it is obviously trivial, so we can regard (and will regard) $c^{n+1}(f)$ as a relative cochain, i.e., an element of $C^{n+1}(X, A; \pi_n(Y))$.

Lemma (Obstruction Cocycle) . *The cochain $c^{n+1}(f)$ is a relative cocycle, i.e.,*

$$\delta\big(c^{n+1}(f)\big) = 0.$$

Proof. We must prove that $c^{n+1}(f)(\partial\Delta^{n+2}) = 0$ for any $(n+2)$-simplex Δ^{n+2} in X. Note that the n-skeleton of the simplicial space $\partial\big(\Delta^{n+2}\big) \approx \mathbb{S}^{n+1}$ is, obviously, an $(n-1)$-connected space (i.e., all its homotopy groups up to the $(n-1)$st are trivial), because it is in fact homotopy equivalent to the sphere $\mathbb{S}^{n+1}$ with several punctures in it (at the baricenters of the $(n+1)$-faces of Δ^{n+2}).

Let $\partial(\Delta^{n+2}) = \sum \Delta_i^{n+1}$. The element $c^{n+1}(f)(\Delta_i^{n+1})$ of the group $\pi_n(X)$ corresponds to the map $f : \partial\big(\Delta^{n+2}\big) \to X$. Denote by B^n the n-skeleton of $\partial(\Delta^{n+2})$. Let α_i be the element of $\pi_n(B^n)$ corresponding to the orientation-preserving homeomorphism $\mathbb{S}^n \to \partial(\Delta_i^{n+1})$. Obviously, the homomorphism

$$f_* : \pi_n(B^n) \to \pi_n(X)$$

takes α_i to $c^{n+1}(f)(\partial\Delta_i^{n+1})$.

Since the space B^n is $(n-1)$-connected, the Hurewicz homomorphism $h : \pi_n(B^n) \to H_n(B^n)$ is an isomorphism (by the Hurewicz Theorem). Hence we can consider he sequence of homomorphisms

$$H_n(B^n) \xrightarrow{h^{-1}} \pi_n(B^n) \xrightarrow{f_*} \pi_n(X);$$

Their composition $f_* \circ h^{-1}$ takes the homology class determined by the cycle $\sum \partial(\Delta_i^{n+1}) =: z$ to $c^{n+1}(f)(\partial\Delta_i^{n+1})$. But

$$\sum \partial(\Delta_i^{n+1}) = \partial\partial\Delta^{n+2} = 0,$$

i.e., the cycle z is zero and therefore its image by the homomorphism $f_* \circ h^{-1}$ is also zero as claimed. □

11.2. The obstruction cohomology class

The obstruction cocycle $c^{n+1}(f)$ determines a cohomology class that we denote by $\Gamma^{n+1}(f)$.

Theorem (Eilenberg) *The map* $f : \widehat{X}^n \to Y$ *can be extended to* $\widehat{X}^{n+1}$ *if and only if* $\Gamma^{n+1}(f) = 0$.

We omit the detailed proof of this theorem, which is rather straightforward. Indeed, in order to extend the map f, we must be able to extend it to each $(n+1)$-simplex from its boundary (on which f is already defined). Roughly speaking, the condition $\Gamma = 0$ says that the image of the boundary of each such simplex is homotopy trivial in Y, so that the extension of the map to the $(n+2)$-simplex is possible.

11.3. The Hopf–Whitney Theorem

The homotopy classification of maps from one topological space to another is one of the fundamental problems of topology. We denote by $[X, Y]$ the set of homotopy classes of maps from X to Y. The next theorem gives a solution of the homotopy classification problem for a wide class of spaces in terms of cohomology of the source space whose coefficients are in the homotopy group of the target space.

Theorem (Hopf–Whitney). *For any* n*-dimensional simplicial space* X *and any* $(n-1)$*-connected space* Y*, the following bijection exists:*

$$\boxed{\big[X, Y\big] \longleftrightarrow H^n(X; \pi_n(Y)).}$$

For the particular case in which $Y = \mathbb{S}^n$, this theorem classifies up to homotopy maps of simplicial complexes to the n-sphere: they correspond bijectively to elements of $H^n(X; \mathbb{Z})$ because $\pi_n(\mathbb{S}^n) = \mathbb{Z}$. Another consequence of this theorem is the following nice geometrical interpretation of the nth cohomology group of an arbitrary n-dimensional simplicial space.

Corollary. *Any element of the cohomology group* $H^n(X, \mathbb{Z})$*, where* X *is an* n*-dimensional simplicial space, can be realized by maps to the sphere, i.e., it can be represented in the form* $f^*(s)$*, where* s *is the generator of the group* $H^n(\mathbb{S}^n; \mathbb{Z})$ *and* $f : X \to \mathbb{S}^n$ *is a map unique up to homotopy.*

11.4. The Eilenberg Mac Lane spaces $K(\pi, n)$

These are spaces whose topology is concentrated, in a certain homotopic sense, in a single dimension. They possess some beautiful properties and are useful in homotopy classification problems as well as in the construction of the s–called cohomology operations. By definition, an *Eilenberg–MacLane space* $K(\pi, n)$ is a topological space X satisfying the condition

$$\pi_k(X) = \begin{cases} \pi & \text{if } k = n; \\ 0 & \text{if } k = n. \end{cases}$$

Examples:

- the circle $\mathbb{S}^1$ is a $K(\mathbb{Z}, 1)$ space;
- any surface M^2 other than $\mathbb{S}^2$ and $\mathbb{R}P^2$ is a $K(\pi_1(M^2), 1)$ space;
- $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2, 1)$ space;
- $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}_2, 2)$ space;
- the infinite-dimensional lens space L_m^∞ is a $K(\mathbb{Z}_m, 1)$ space.

The validity of these specific examples will be discussed in the exercise class. The existence of many other $K(\pi, n)$ spaces will be established in the next theorem.

Theorem (Existence of $K(\pi, n)$ spaces). *For any finitely presented group π and any n there exists a $K(\pi, n)$ space.*

Proof. The proof is by a direct construction, in which we first construct a space with zero homotopy groups in all dimensions up to $n-1$, then supply it with the necessary n-homotopy group (isomorphic to π), and finally kill all the higher homotopy inductively by "chasing them away to infinity".

Let $\langle g_1, \dots, g_r | R_1 = \dots = R_s = 1 \rangle$ be a presentation of π. Denote by K^n the wedge of r copies of the (triangulated) n-sphere. Take s copies of the $(n+1)$-disk and glue it to the wedge of spheres in accordance to the relations $R_1, \dots, R_s$, obtaining a simplicial space that we denote by K^{n+1}. Clearly, $H_n(K^{n+1}) \cong \pi$. By the Hurewicz Theorem, we have $\pi_n(K^{n+1}) \cong \pi$.

However, our construction is not finished because K^{n+1} can have nontrivial higher (than n) homotopy groups. We kill these groups inductively by gluing disks onto their generators. $\square$

Theorem (Uniqueness of $K(\pi, n)$ spaces). *Two $K(\pi, n)$ spaces with the same π and n are homotopy equivalent.*

Proof. It suffices to prove that any $K(\pi, n)$ space is homotopy equivalent to the one constructed in the proof of the Existence Theorem. This is done inductively on the dimension of the skeletons of the given space, the key point is the application of obstruction theory, which is easy because the corresponding obstructions vanish. $\square$

Theorem (Maps to $K(\pi, n)$ spaces). *If Y is a simplicial $K(\pi, n)$ space and X is any simplicial space, then there is a bijection*

$$\big[X, Y\big] \longleftrightarrow H^n(X; \pi).$$

Proof. Let us denote by $F_\pi \in H^n(X, \pi)$ the fundamental class of the space Y (i.e., the class corresponding to the inverse Hurewicz isomorphism $h^{-1} : H_n(X) \to \pi_n(X)$, see the proof of the Existence Theorem). To any map $f : X \to Y$ we assign the cohomology class $f^*(F_\pi) \in H^n(X; \pi)$; clearly this assignment depends only on the homotopy class of f. Its bijectivity follows from the Hopf–Whitney Theorem. The details are left to the reader. □

Corollary. *Maps from one Eilenberg-MacLane space to another (with the same n but possibly different π's) are classified up to homotopy by the homomorphisms of their groups:*

$$\big[K(\pi, n), K(\pi', n)\big] \longleftrightarrow \mathrm{Hom}(\pi, \pi').$$

Proof. The so-called universal coefficients formula, which will be discussed in the next lecture, implies that

$$H^n(K(\pi, n); \pi') \cong \mathrm{Hom}(H_n(K(\pi, n); \pi')),$$

so that the corollary follows from the previous theorem. □

11.5. Problems

11.1. Prove that $\mathbb{S}^p \times \mathbb{S}^q = (\mathbb{S}^p \wedge \mathbb{S}^q) \bigcup_f \mathbb{D}^{p+q}$, where f is some map $f : \mathbb{S}^{p+q} \to \mathbb{S}^p \wedge \mathbb{S}^q$ non homotopic to the constant map.

11.2. In the notation of Sec.11.2 of the lecture, let $f, g : \widehat{K}^n \to Y$ be two maps that coincide on $\widehat{K}^{n-1}$. Denote by $d^n(f,g) \in C^n(K^n, A; \pi_n(Y))$ the cochain (called *distinguishing*) defined on a simplex Δ^n as follows: take two copies of Δ^n, glue then along their boundary, map one of the copies via f and the other via g to Y, thus determining an element of $\pi_n(Y)$. Prove that

$$d^n(f,g) = c^n(g) - c^n(f).$$

Use this fact to prove the Eilenberg Theorem.

11.3. Prove that $d^n(f,g) + d^n(g,h) + d^n(h,f) = 0$, where d^n is the distinguishing cochain defined in Problem 11.2.

11.4. Prove that $\mathbb{S}^1$ is a $K(\mathbb{Z}, 1)$ space.

11.5. Let M^2 be a closed surface other than $\mathbb{S}^2$ or $\mathbb{R}P^2$. Prove that M^2 is a $K(\pi_1(M^2), 1)$ space.

11.6. Prove that $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}_2, 1)$ space.

11.7. Prove that $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ space.

11.8. Prove that L_m^∞ is a $K(\mathbb{Z}_m, 1)$ space, where the *infinite-dimensional lens space* L_m^∞ is defined as follows: consider $\mathbb{S}^\infty$ as the subspace of $\mathbb{C}^\infty$ of all points $(z_1, z_2, \dots)$ such that $\sum |z_i^2| = 1$; introduce the equivalence relation $(z_1, z_2, \dots) \sim (\varepsilon z_1, \varepsilon z_2, \dots)$, where $\varepsilon = \exp(2\pi i/m)$; then $L_m^\infty := \mathbb{S}^\infty/\sim$.

11.9. Prove that

$$H_n(L^\infty) = \begin{cases} \mathbb{Z}_m & \text{for odd } n; \\ 0 & \text{for even } n. \end{cases}$$

11.10. Let M^3 be a closed 3-manifold such that $\pi_1(M^3)$ is infinite and $\pi_2(M^3) = 0$. Prove that M^3 is a $K(\pi, 1)$ space for some π.

11.11. Let X be a finite-dimensional simplicial $K(\pi, 1)$ space. Prove that π has no finite-order elements.

Lecture 12

VECTOR BUNDLES AND G-BUNDLES

In this lecture, we study and classify vector bundles (a key notion in differential topology, the main example being the tangent bundle of a smooth manifold). and principal G-bundles (which play an important role in geometric topology, K-theory, and theoretical physics). Vector bundles are classified over paracompact spaces by using the partition of unity, a generalization of the Feldbau Theorem, and the Gauss map to the canonical Grassmann bundle, while G-bundles are classified via the beautiful Milnor construction and the notion of classifying space of a topological group.

12.1. The category of vector bundles

A *vector bundle* $p : E \to B$ is a (locally trivial) fiber bundle (see Sec.2.3) whose fiber F is a vector space of fixed dimension n. A *morphism* of vector bundles $(\varphi, \Phi) : p_1 \to p_2$ is a pair of maps $\Phi : E_1 \to E_2$ and $\varphi : B_1 \to B_2$ such that $\varphi \circ p_1 = \Phi \circ p_2$ and the restriction of Φ to any fiber, i.e.,

$$\Phi|_{p_1^{-1}(b)} : p_1^{-1}(b) \to p_2^{-1}(\varphi(b))$$

is a linear map for any $b \in B_1$.

Given a vector bundle $p : E \to B$ and a subset $X \subset B$, one defines the *restriction* of p to B_1, denoted $p|_X$ in the natural way; more generally, if $f : X \to B$ is a map, then the *pullback* $f^*(p) : E_1 \to X$ of p to X is defined as follows:

$$E_1 := \big\{(b, e) \in X \times E : f(b) = p(e)\big\} \quad \text{and} \quad f^*(p)(b, e) := b.$$

In that situation, there is a canonical morphism $f^*(p) \to p$ given by the formulas $\varphi(b) = f(b)$, $\Phi((b, e)) = e$.

A morphism of vector bundles $\varphi : p_1 \to p_2$ is said to be an *isomorphism* if there exists a morphism $\psi : p_2 \to p_1$ such that we have $\Psi \circ \Phi = \mathrm{id}_{B_1}$ and $\Phi \circ \Psi = \mathrm{id}_{B_2}$.

Lemma. *A morphism of fiber bundles over the same base is an isomorphism if and only if its restriction to any fiber is an isomorphism of linear spaces.*

The proof is an exercise.

Vector bundles are usually considered over paracompact bases (because partition of unity is needed to develop the theory). Vector bundles over all paracompact spaces form a category that will be denoted by $\mathcal{V}$ect. We shall also consider the categories of vector bundles over fixed paracompact bases; they are denoted $\mathcal{V}\mathrm{ect}(B)$, where B is the (paracompact) base, and $\mathcal{V}\mathrm{ect}_k(B)$, when the dimension k of the fiber is fixed.

Examples:

• the trivial bundles, or product bundles $pr_1 : \mathbb{R} \times B \to B$, $pr_1(r, b) = b$;

• the tangent bundle $\tau : TM^n \to M^n$ of a smooth manifold;

• the normal bundle $\nu : NM^n \to M^n$ of a smooth manifold smoothly embedded in $\mathbb{R}^N$;

• the *canonical Grassmann bundle* $\gamma_k^m : E_k^m \to G_k^m$, where G_k^m is the Grassmann manifold (i.e., the set, supplied with the natural topology, whose points are the k-dimensional linear subspaces L of the vector space $\mathbb{R}^m$),

$$E_k^m := \big\{(L, r) \in G_k^m \times \mathbb{R}^m : r \in L\big\},$$

and γ_k^m is the natural projection $(L, r) \mapsto L$. There are obvious inclusions $G_k^m \subset G_k^{m+1}$ which allow to define G_k^∞ (called the *Grassmannian*) and γ_k^∞ by passing to the inductive limit.

12.2. Classification of vector bundles over a given base

In this section, we study the category $\mathcal{V}\mathrm{ect}_k(B)$ of k-vector bundles over a fixed paracompact base B. The main result is the following.

Theorem (Classification). *The isomorphism classes of k-vector bundles $p : E \to B$ over a paracompact base B are in bijective correspondence with the homotopy classes of maps of the base to the Grassmannian G_k^∞,*

$$\boxed{\big[B, G_k^\infty\big] \longleftrightarrow \mathcal{V}\mathrm{ect}_k(B)}\,,$$

where the correspondence assigns to each map $f : B \to G_k^\infty$ the pullback bundle $f^*(\gamma^{\infty_k})$.

To prove the theorem, we must establish three facts.

(i) The correspondence is well defined, i.e., any two homotopic maps $f, g : B \to G_k^\infty$ have isomorphic pullbacks $f^*(\gamma_k^\infty) \cong g^*(\gamma_k^\infty)$.

(ii) The correspondence is injective, i.e., if the bundles $f^*(\gamma^{\infty_k})$ and $g^*(\gamma^{\infty_k})$ are isomorphic, then the maps f and g, $f, g : B \to G_k^\infty$, are homotopic.

(iii) The correspondence is surjective, i.e., for any bundle $p \in \mathcal{V}\mathrm{ect}_k(B)$ there is a map $f : B \to G_k^\infty$ such that the pullback bundle $f^*(\gamma^{\infty_k})$ is isomorphic to p.

Let us comment on the proofs of these facts.

Item (i) is a particular case of the following more general theorem.

Theorem (Homotopy Invariance). *For any vector bundle $\gamma : E \to B$ with paracompact base B, homotopic maps f, g of a paracompact space B_1 to B have isomorphic pullback bundles $f^*(\gamma) \cong g^*(\gamma)$.*

This theorem has a corollary, which is a far-going generalization of the Feldbau Theorem (Lecture 2, Sec.2.3).

Corollary (Triviality). *Any vector bundle over a contractible paracompact space is trivial.*

The proof of the Homotopy Invariance Theorem is based on the following lemma.

Lemma (Multiplication by $[0,1]$). *For any vector bundle p with basis $B \times [0,1]$ there exists an open covering $\{U_\alpha\}$ of B such that the restriction of the bundle to each of the sets $U_\alpha \times [0,1]$ is trivial.*

Item (ii) is proved by using a version of the *Gauss map*, which we first define in the case of a smooth manifold M^k embedded in $\mathbb{R}^N$, as the vector bundle $\gamma_k^N : TM^k \to G_k^N$ obtained by the parallel transport of all the tangent planes $T_x(M^k)$, $x \in M^k$, to the origin 0 (so that x is taken to 0).

In the general case, the construction is somewhat more difficult, and requires using the infinite-dimensional Grassmannian and the notion of *Gauss inclusion* of a vector bundle $p : E \to B$, defined as any map $g : E \to \mathbb{R}^\infty$ which is a linear monomorphism on the fibers.

The proof of (ii) is based on the two following lemmas.

Lemma (Local Triviality). *For any vector bundle $p : E \to B$ with paracompact base B there exists a countable open covering $\{U_i\}$ such that the restriction of p to each U_i is trivial.*

Lemma (Gauss Inclusion). *For any vector bundle $p : E \to B$ with paracompact base B there exists a Gauss inclusion $g : E \to \mathbb{R}^\infty$.*

Item (iii) of the classification theorem is proved, in the general case, by using a sophisticated argument based on presenting $\mathbb{R}^\infty$ as the linear sum

$\mathbb{R}^{ev} \oplus \mathbb{R}^{od}$ of the subspaces generated by the basis vectors with even and odd coordinates. For the particular case in which the given bundle is the tangent bundle τ of a smooth manifold M^k, the proof is easy: we embed M^k in some $\mathbb{R}^N$, regard G_k^N as consisting of k-dimensional linear subspaces of $\mathbb{R}^N$ and apply the Gauss map, assigning to each point $x \in M^k$ the linear subspace obtained by parallel transport of the tangent plane $T_x M^k$ to the origin.

A good reference for the proofs is the book *Fiber Bundles* by D.Husemoller (Rassloennye prostranstva, M., Nauka, 1970).

12.3. The category of G-bundles

A *topological group* is a topological space G with a group structure compatible with its topology, i.e., such that the maps $G \times G \to G$, $(s,t) \mapsto st$, and $G \to G$, $s \mapsto s^{-1}$, are continuous. We say that topological group G *acts from the right* on a topological space X if we are given a continuous map $X \times G \to X$, $(x,g) \mapsto xg$ (xg is called the *image* of the point x under the action of the element g) such that

- $x(gh) = (xg)h$ for all $x \in X$ and all $g, h \in G$;
- $x1 = x$ for all $x \in X$, 1 being the unit (neutral element of G).

A left action of a topological group G on a topological space X is defined similarly. If G is Abelian, then left actions are also right actions and vice versa (but the notations remain different).

Examples:

(1) the (natural) left actions of $GL(n)$ and of $O(n)$) on $\mathbb{R}^n$;

(2) the (natural) right action of the group of orthogonal matrices $O(k)$ on the *Stiefel manifold* V_k^n of orthonormal k-frames in $\mathbb{R}^n$;

(3) the left (=right) action of the nonzero real numbers on $\mathbb{R}^n$ by multiplication of the coordinates of points by these numbers.

A *G-bundle* $p : E \to B$ is the projection of a topological space E, supplied with a right action of a topological group G, onto the orbit space $B = E/G$ of this action. Note that unlike vector bundles, G-bundles are not necessarily locally trivial. Thus a G-bundle doesn't have to be a fiber bundle, and of course for a fiber bundle p there doesn't have to exist a group G such that p is a G-bundle.

A morphism of G-bundles is defined in the natural way. The class of all G-bundles forms a category denoted $\mathcal{B}\mathrm{un}_G$; by fixing the base B, we obtain one of its subcategories, denoted $\mathcal{B}\mathrm{un}_G(B)$.

A G-bundle $p : E \to B$ is called *principal* if its fiber $F = p^{-1}(b)$ is (homeomorphic to) the group G for all $b \in B$. The class of all principal G-bundles forms a category denoted $\mathcal{PB}\mathrm{un}_G$; by fixing the base B, we obtain one of its subcategories, denoted $\mathcal{PB}\mathrm{un}_G(B)$.

Examples.

(1) The identification of opposite points on the n-dimensional sphere is a principal $\mathbb{Z}_2$ bundle over $\mathbb{R}P^2$.

(2) The natural projection of the Stiefel manifold V_k^n onto the Grassmann manifold G_k^n is a principal $O(k)$-bundle.

12.4. The Milnor construction

Let G be any topological group; denote by $E_G(n) := G * G * \cdots * G$ (n factors) the n-fold iterated join of the group G with itself. Obviously

$$G \subset E_G(2) \subset E_G(3) \subset \cdots \subset E_G(n) \subset \cdots \subset E_G,$$

where E_G is the inductive limit of $E_G(n)$ as $n \to \infty$.

Consider the action of G in E_G by right shifts. The corresponding bundle

$$\omega_G : E_G \to B_G = E_G/G$$

is called the *universal G-bundle*, its base is called the *classifying space* of the group G. In a similar way, one defines the bundle $\omega_G^n : E_G^n \to B_G^n$ called (briefly) the *n-universal G-bundle*, its base is called the *n-classifying space* of the group G (in less shortened form, the *classifying space up to dimension n*).

Examples:

(1) The classifying space of the group $\mathbb{S}^1$ is $\mathbb{C}P^\infty$ and $E_{\mathbb{S}^1} = \mathbb{S}^\infty$. The k-classifying space of $\mathbb{S}^1$ is $\mathbb{C}P^k$ and $E_{\mathbb{S}^1}^k = \mathbb{S}^{2k+1}$.

(2) The classifying space of the group $\mathbb{Z}_2$ is $\mathbb{R}P^\infty$ and $E_{\mathbb{Z}_2} = \mathbb{S}^\infty$. The k-classifying space of $\mathbb{Z}_2$ is $\mathbb{R}P^k$ and $E_{\mathbb{S}^1}^k = \mathbb{S}^k$.

12.5. Classification of principal G-bundles

The classification of principal G-bundles is similar, but more complicated, than the one for vector bundles. The role of the canonical Grassmann bundle here is played by Milnor's universal G-bundle $\omega_G : E_G \to B_G$, and the main idea is the same: to take the pullback $f^*(\omega_G)$ of the universal bundle for maps $f : B \to B_G$.

Theorem (Homotopy Invariance). *If two maps $f, g : B \to B_G$ of the same space B to the classifying space B_G are homotopic, then the corresponding pullbacks $f^*(\omega_G)$ and $g^*(\omega_G)$ are isomorphic.*

Corollary (Triviality). *Any principal bundle over a contractible paracompact space is trivial.*

Theorem (Principal Bundle Classification). *For any paracompact space B, the assignment $[B, B_G] \ni f \mapsto f^*(\omega_G)$ determines a bijection*

$$\boxed{[B, B_G] \longleftrightarrow \mathcal{PB}un_G(B)}$$

between the homotopy classes of maps of B into the classifying space B_G and the isomorphism classes of principal G-bundles over B.

We will not prove this theorem in the present course (see the book by Hosemueller, *loc.cit*).

12.6. Bundles associated with principal G-bundles

Given a topological group G, a principal G-bundle $\xi : E \to B$, and a topological space F with a left action of G on it, one can construct in a canonical way the *associated a fiber bundle* denoted by $\xi[F]; E[F] \to B$ by "replacing the fiber G of ξ by F"; more precisely, $\xi[F]$ is defined as follows: in the space $E \times F$, consider the right action of G defined by $(e, f)g = (eg, g^{-1}f)$, and denote

$$E[F] := (E \times F)/G, \quad \text{and} \quad \xi[F] := \xi \circ \mathrm{pr}_1,$$

where pr_1 is the projection $\mathrm{pr}_1(e, f) = e$.

Bundles associated with principal G-bundles are sometimes called *bundles with fiber F and structural group G* or briefly (G, F)*-bundles.* Note that such bundles are not necessarily locally trivial. A morphism of two bundles $\xi[F]$ and $\xi'[F]$ is defined as a pair of maps $\bar{\varphi} : B \to B'$, $\bar{\Phi} : E[F] \to E'[F]$ forming a commutative square with the bundle projections provided there exists a morphism of the principal G-bundles $(\Phi, \varphi) : (E, B) \to (E', B')$ such that $\bar{\Phi}(e, f) = \Phi(e)$ and $\bar{\varphi}(b) = \varphi(b)$.

Thus, if a left action of a topological group G on a topological space F is given, then we can consider the category $\mathcal{B}\mathrm{un}_{(G,F)}$ and, if the (paracompact) base B is fixed, the category $\mathcal{B}\mathrm{un}_{(G,F)}(B)$.

The very rich theory of these categories is beyond the scope of this course.

Lecture 13

MISCELLANY

In this very eclectic lecture, I have gathered some important constructions and theorems (presented here without proofs) which should appear in any serious introductory course in algebraic topology, but which I was unable to treat in detail in the previous lectures.

13.1. The functors Hom, Tor, and Ext

Given two Abelian groups A, B, the group $\mathrm{Hom}(A, B)$ is defined as the set of homomorphisms of A to B with the natural group structure. Using Hom, we define the groups $\mathrm{Tor}(A, B)$ and $\mathrm{Ext}(A, B)$ as follows. Let

$$0 \longrightarrow R \xrightarrow{\ i\ } F \xrightarrow{\ p\ } A \longrightarrow 0,$$

be the exact sequence which is the free resolution of the group A (so F is the free group in the generators of A and R is the free group generated by the relations of A). Tensoring this sequence with B and taking $\mathrm{Hom}(\cdot, B)$ of this sequence, we obtain two exact sequences

$$R \otimes B \xrightarrow{\ i\otimes 1\ } F \otimes B \xrightarrow{\ p\otimes 1\ } A \otimes B \longrightarrow 0,$$

$$\mathrm{Hom}(R, B) \xleftarrow{\ \bar{i}\ } \mathrm{Hom}(F, B) \xleftarrow{\ \beta\ } \mathrm{Hom}(A, B) \longleftarrow 0,$$

Then we set $\mathrm{Tor}(A, B) := \mathrm{Ker}(i \otimes 1)$ and $\mathrm{Ext}(A, B) := \mathrm{Coker}\big(\bar{i}\big)$. It is not difficult to check that Tor and Ext are well defined, i.e., do not depend on the choice of the free resolution.

Examples.

(1) $\mathrm{Tor}(\mathbb{Z}, B) = 0$, $\mathrm{Ext}(\mathbb{Z}, B) = 0$;

(2) $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}) = 0$, $\mathrm{Ext}(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$;

(3) $\mathrm{Tor}(\mathbb{Z}_p, \mathbb{Z}_q) = \mathbb{Z}_{(p,q)}$, $\mathrm{Ext}(\mathbb{Z}_p, \mathbb{Z}_q) = \mathbb{Z}_{(p,q)}$, where (p, q) is the greatest common divisor of p and q.

(4) $\mathrm{Ext}(A, T) = T$ if $A = \mathbb{Z}^k \oplus T$, where T is a finite group.

(5) $\mathrm{Tor}(\mathbb{G}, B) = 0$, $\mathrm{Ext}(A, \mathbb{G}) = 0$ if $\mathbb{G}$ is the additive group of one of the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

(6) $\mathrm{Tor}(A, B) = 0$ if A is an Abelian group such that $\mathrm{Ker}(\mu_n) = 0$ for any $n \in \mathbb{N}$, where the homomorphism μ_n is given by $\mu_n(a) = na$.

13.2. Universal coefficient formula

The universal coefficient formula allows to express the (co)homology groups with any coefficient group G in terms of its (co)homology with integer coefficients. We have the following theorem.

Theorem (Universal Coefficients) *For any Abelian group G and any simplicial space X, there exist two exact sequences*

$$0 \longrightarrow H_k(X) \otimes G \longrightarrow H_k(X;G) \longrightarrow \mathrm{Tor}(H_{k-1}(X), G) \longrightarrow 0,$$

$$0 \longleftarrow \mathrm{Hom}(H_k(X), G) \longleftarrow H^k(X;G) \longleftarrow \mathrm{Hom}(H_{k-1}(X), G) \longleftarrow 0;$$

these sequences split and therefore

$$\boxed{H_k(X;G) \cong \big(H_k(X) \otimes G\big) \oplus \mathrm{Tor}\big(H_{k-1}(X), G\big)}$$

$$\boxed{H^k(X;G) \cong \mathrm{Hom}\,\big(H_k(X), G\big) \oplus \mathrm{Ext}\big(H_{k-1}(X), G\big)}.$$

Corollary. *If $\mathbb{G}$ is the additive group of one of the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, then*

$$H_k(X;G) \cong H_k(X) \otimes G,$$

and

$$H^k(X;G) \cong \mathrm{Hom}\,\big(H_k(X), G\big).$$

13.3. Künneth formula

The Künneth formula expresses the (integer) homology of the Cartesian product of two simplicial spaces in terms of the homology of the factors, namely

$$\boxed{H_k(X \times Y) \cong \bigoplus_{l+m=k} \big(H_l(X) \otimes H_m(Y)\big) \oplus \bigoplus_{l+m=k-1} \mathrm{Tor}(H_l(X), H_m(Y))}.$$

For the case in which the coefficient group is $\mathbb{G}$, the additive group of one of the fields $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, then we have the simpler relation

$$H_k(X \times Y) \cong \bigoplus_{l+m=k} \big(H_l(X) \otimes H_m(Y)\big).$$

13.4. Alexander duality

Alexander duality expresses the reduced (co)homology of the complement to a submanifold in $\mathbb{S}^n$ in terms of the manifold's (co)homology.

Theorem (Alexander Duality). *If M is a submanifold (not necessarily smooth) in the n-sphere $\mathbb{S}^n$, then, for any k,*

$$\boxed{\widetilde{H}^k(M) \cong \widetilde{H}_{n-k-1}(\mathbb{S}^n \setminus M),\ \widetilde{H}_k(M) \cong \widetilde{H}^{n-k-1}(\mathbb{S}^n \setminus M)}.$$

Corollary (Jordan-Brouwer Theorem). *Any $(n-1)$-sphere $\mathbb{S}^{n-1}$ embedded in $\mathbb{R}^n$ splits $\mathbb{R}^n$ into two connected components.*

13.5. The Poincaré–Hopf theorem

This generalizes Poincaré's theorem on vector fields on surfaces.

Theorem (Poincaré–Hopf). *The index of a generic vector field on a closed smooth orientable manifold equals the manifold's Euler characteristic.*

13.6. Čech homology

Čech homology, which is defined for arbitrary topological spaces, was invented (before singular homology) by P.S.Alexandrov (and not by Čech). We do not describe it in detail here, only indicating the two main ideas underlying its definition.

Given an arbitrary covering of $\omega = \{U_\alpha\}$ of a topological space X, let us define the *nerve* N_ω of this covering as the family of simplices corresponding to this covering: the 0-simplices (vertices) are the open sets of the covering ω, the 1-simplices are pairs $(U_{\alpha_1}, U_{\alpha_2})$ such that $U_{\alpha_1} \cap U_{\alpha_2} \neq \varnothing$, the 2-simplices are triplets of vertices having at least one common point, etc.

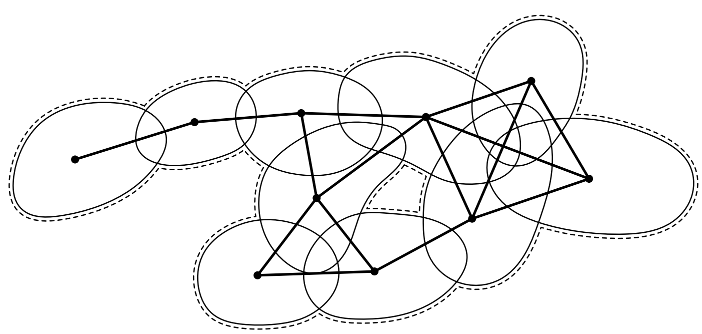


FIGURE 13.1 Nerve of a covering

Chain groups (say with integer coefficients) $C_k(N_\omega)$ are defined as the sets of all linear combinations of k-simplices, the boundary operator is defined as in simplicial homology, and it yields the homology groups $H_k(N_\omega)$.

It should be intuitively clear that if X is a "nice" enough and the covering ω is "sufficiently fine", then the nerve N_ω is a "good approximation" of the topology of X so that $H_k(N_\omega)$ is a good approximation of the k-homology of the space X.

Alexandrov's second main idea, roughly speaking, is based on the fact that if one covering is inscribed in another, there is a natural projection of the homology of the inscribed nerve onto the homology of the other nerve, and the definition of the homology groups is obtained by taking the projective limit of these groups.

It can be proved that the Čech homology functor satisfies the Steenrod-Eilenberg axioms and therefore coincides, say, with the singular homology functor.