Lecture 7

THE POINCARÉ DISK MODEL OF HYPERBOLIC GEOMETRY

In this lecture, we begin our study of the most popular of the non-Euclidean geometries – hyperbolic geometry, concentrating on the case of dimension two. We avoid the intricacies of the axiomatic approach (which will only be sketched in Chapter 10) and define hyperbolic plane geometry via the beautiful Poincaré disk model, which is the geometry of the disk determined by the action of a certain transformation group acting on the disk (namely, the group generated by reflections in circles orthogonal to the boundary of the disk).

In order to describe the model, we need some facts from Euclidean plane geometry, which should be studied in high school, but in most cases unfortunately aren't. So we begin by recalling some properties of inversion (which will be the main ingredient of the transformation group of our geometry) and some constructions related to orthogonal circles in the Euclidean plane. We then establish the basic facts of hyperbolic plane geometry and finally digress (following Poincaré's argumentation from his book *Science et Hypothèse* [??]) about epistomological questions relating this geometry (and other geometries) to the physical world.

7.1. Inversion and orthogonal circles

7.1.1. *Inversion and its properties.* The main tool that we will need in this lecture is inversion, a classical transformation from elementary plane geometry. Denote by $\mathcal{R}$ the plane $\mathbb{R}^2$ with an added extra point (called the *point at infinity* and denoted by ∞). The set $\mathcal{R} := \mathbb{R}^2 \cup \infty$ can also be interpreted as the complex numbers $\mathbb{C}$ with the "point at infinity" added; it is then called the *Riemann sphere* and denoted by $\overline{\mathbb{C}}$.

An *inversion* of center $O \in \mathbb{R}^2$ and radius $r > 0$ is the transformation of $\mathcal{R}$ that maps each point M to the point N on the ray OM so that

$$\boxed{|OM| \cdot |ON| = r^2} \tag{7.1}$$

and interchanges the points O and ∞. Sometimes inversions are called *reflections* with respect to the *circle of inversion*, i.e., the circle of radius r centered at O.

There is a simple geometric way of constructing the image of a point M under an inversion of center O and radius r: draw the circle of inversion, lower the perpendicular to OM from M to its intersection point T with the circle and construct the tangent to the circle at T to its intersection point N with the ray OM; then N will be the image of M under the given inversion. Indeed, the two right triangles OMT and OTN are similar (they have a common acute angle at O), and therefore

$$\frac{|OM|}{|OT|} = \frac{|OT|}{|ON|},$$

and since $|OT| = r$, we obtain (7.1).

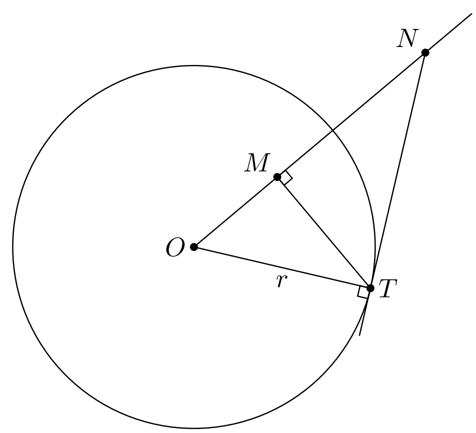

Figure 7.1. Inversion $|OM| \cdot |ON| = r^2$

If the extended plane $\mathcal{R}$ is interpreted as the Riemann sphere $\overline{\mathbb{C}}$, then an example of an inversion (of center O and radius 1) is the map $z \mapsto 1/\overline{z}$, where the bar over z denotes complex conjugation.

It follows immediately from the definition that inversions are bijections of $\mathcal{R} = \overline{\mathbb{C}}$ that leave the points of the circle of inversion in place, "turn the circle inside out" in the sense that points inside the circle are taken to points outside it (and vice versa), and are *involutions* (i.e., the composition of an inversion with itself is the identity). Further, inversions possess the following important properties.

(i) *Inversions map any circle or straight line into a circle or straight line.* In particular, lines passing through the center of inversion are mapped to themselves (but are "turned inside out" in the sense that O goes to ∞ and vice versa, while the part of the line inside the circle of inversion goes to the outside part and vice versa); circles passing through the center of inversion are taken to straight lines, while straight lines not passing through the center of inversion are taken to circles passing through that center (see Fig.7.2).

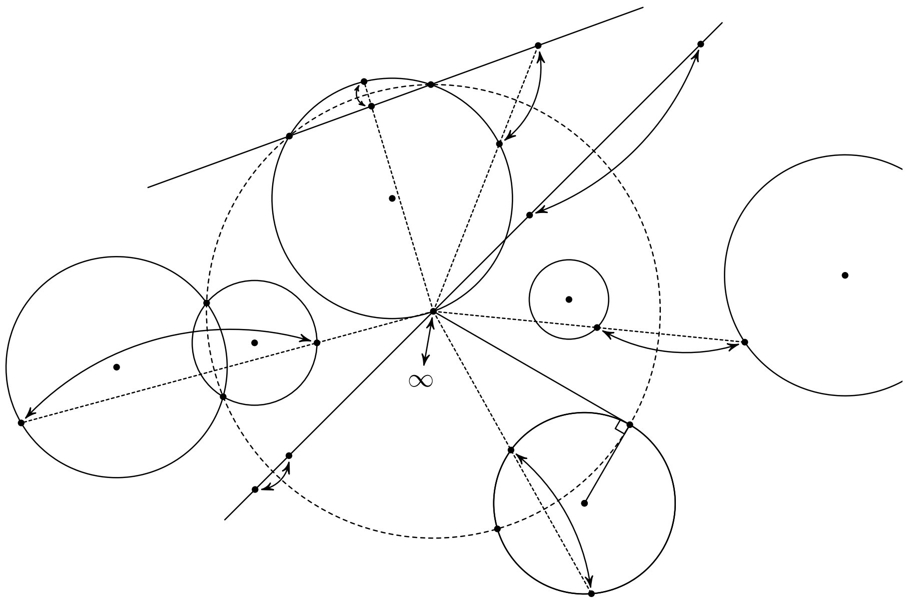

Figure 7.2. Images of circles and lines under inversion

(ii) *Inversions preserve* (*the measure of*) *angles;* here by the measure of an angle formed by two intersecting curves we mean the ordinary (Euclidean) measure of the angle formed by their tangents at the intersection point.

(iii) *Inversions map any circle or straight line orthogonal to the circle of inversion into itself.* Look at Fig.7.3, which shows two orthogonal circles $\mathcal{C}_O$ and $\mathcal{C}_I$ of centers O and I, respectively.

It follows from the definition of orthogonality that the tangent from the center O of $\mathcal{C}_O$ to the other circle $\mathcal{C}_I$ passes through the intersection point T of the two circles. Now let us consider the inversion of center O and radius $r = |OT|$. According to property (iii) above, it takes the circle $\mathcal{C}_I$ to itself; in particular, the point M is mapped to N, the point T (as well as the other intersection point of the two circles) stays in place, and the two arcs of $\mathcal{C}_I$ cut out by $\mathcal{C}_O$ are interchanged. Note further that, vice versa, the inversion in the circle $\mathcal{C}_I$ transforms $\mathcal{C}_O$ in an analogous way.

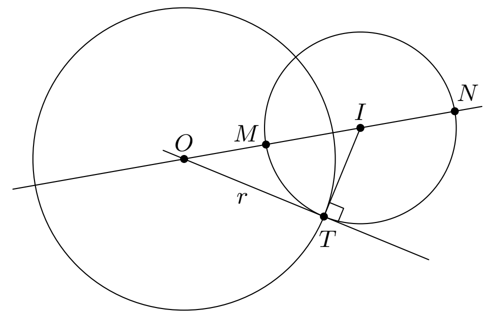

Figure 7.3. Orthogonal circles

The (elementary) proofs of properties (i)–(iii) are left to the reader (see Exercises 7.1–7.3).

7.1.2. *Construction of orthogonal circles.* We have already noted the important role that orthogonal circles play in inversion (see 7.1.1.(iii)). Here we will describe several constructions of orthogonal circles that will be used in subsequent sections.

7.1.3. Lemma. *Let A be a point inside a circle $\mathcal{C}$ centered at some point O; then there exists a circle orthogonal to $\mathcal{C}$ such that the reflection in this circle takes A to O.*

Proof. From A draw the perpendicular to line OA to its intersection T with the circle $\mathcal{C}$ (see Fig.7.4).

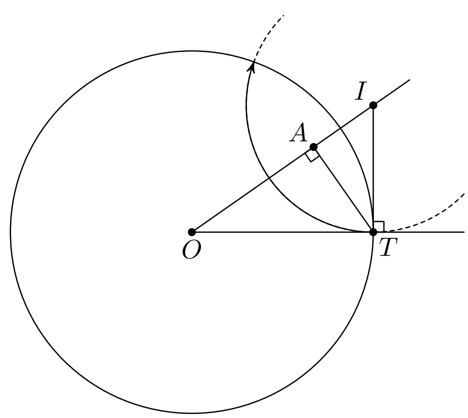

Figure 7.4. Inversion taking an arbitrary point A to O

Draw the tangent to $\mathcal{C}$ at T to its intersection at I with OA. Then the circle of radius IT centered at I is the one we need. Indeed, the similar right triangles OAT and ITO yield $|IA|/|IT| = |IT|/|IO|$, whence we obtain $|IA| \cdot |IO| = |IT|^2$, which means that O is the reflection of A in the circle of radius $|IT|$ centered at I, as required. □

7.1.4. Corollary. (i) *Let A and B be points inside a circle $\mathcal{C}_0$ not lying on the same diameter; then there exists a unique circle orthogonal to $\mathcal{C}_0$ and passing through A and B.*

(ii) *Let A be a point inside a circle $\mathcal{C}_0$ and P a point on $\mathcal{C}_0$, with A and P not lying on the same diameter; then there exists a unique circle orthogonal to $\mathcal{C}_0$ passing through A and P.*

(iii) *Let P and Q be points on a circle $\mathcal{C}_0$ of center O such that PQ is not a diameter; then there exists a unique circle $\mathcal{C}$ orthogonal to $\mathcal{C}_0$ and passing through P and Q.*

(iv) *Let A be a point inside a circle $\mathcal{C}_0$ of center O and $\mathcal{D}$ be a circle orthogonal to $\mathcal{C}_0$; then there exists a unique circle $\mathcal{C}$ orthogonal to both $\mathcal{C}_0$ and $\mathcal{D}$ and passing through A.*

Proof. To prove (i), we describe an effective step-by-step construction, which can be carried out by ruler and compass, yielding the required circle. The construction is shown on Figure 7.5, with the numbers in parentheses near each point indicating at which step the point was obtained.

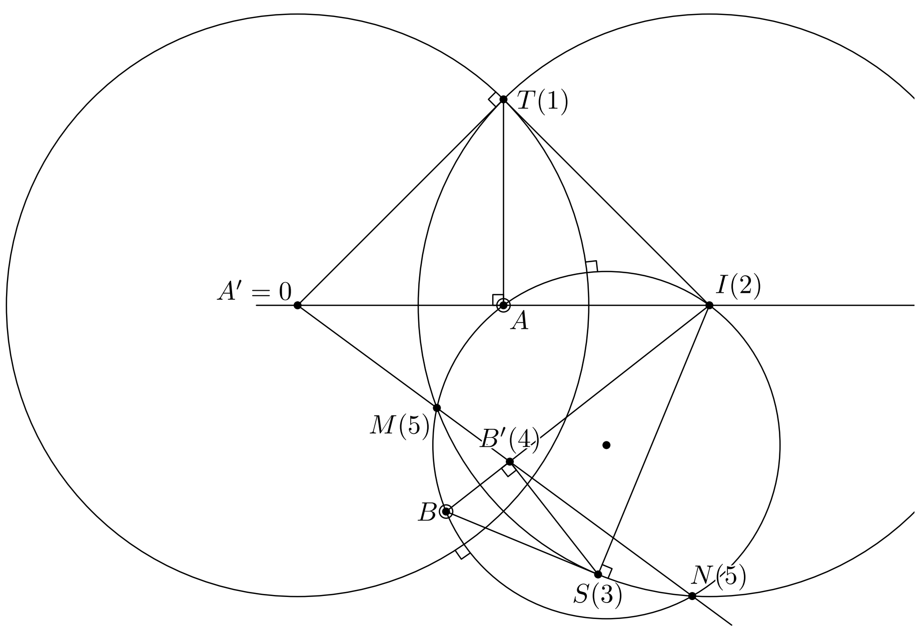

Figure 7.5. Circle orthogonal to $\mathcal{C}_0$ containing A, B

First, we apply Lemma 7.1.3, to define an inversion φ taking A to the center O of the given circle; to do this, we raise a perpendicular from A to OA to its intersection T (1) with $\mathcal{C}$, then draw the perpendicular to OT from T to its intersection I (2) with OA; the required inversion is centered at I and is of radius $|IT|$. Joining B and I, we construct the tangent BS (3) to the circle of the inversion φ and find the image B' (4) of B under φ by dropping a perpendicular from S to IB.

Next, we draw the line $B'O$ and obtain the intersection points M, N of this line with the circle of the inversion φ. Finally, we draw the circle $\mathcal{C}$ passing through the points M, N, I. Then $\mathcal{C}$ "miraculously" passes through A and B and is orthogonal to $\mathcal{C}_0$! Of course, there is no miracle in this: $\mathcal{C}$ passes through A and B because it is the inverse image under φ of the line OB' (see 7.1.1(i)), it is orthogonal to $\mathcal{C}_0$ since so is OB' (see 7.1.1(ii)).

Uniqueness is obvious in the case $A = O$ and follows in the general case by 7.1.1(i)-(ii). □

The proof of (ii) is analogous: we send A to O by an inversion φ, join O and $\varphi(P)$ and continue the argument as above. □

To prove (iii), construct lines OP and OQ, draw perpendiculars to these lines from P and Q respectively and denote by I their intersection point. Then the circle of radius $|IP|$ centered at I is the required one. Its uniqueness is easily proved by contradiction. □

To prove (iv), we again use Lemma 7.1.3 to construct an inversion φ that takes $\mathcal{C}_0$ to itself and sends A to O. From the point O, we draw the (unique) ray $\mathcal{R}$ orthogonal to $\varphi(\mathcal{L})$. Then the circle $\varphi^{-1}(\mathcal{R})$ is the required one. □

7.2. Definition of the disk model

7.2.1. The disk model of the *hyperbolic plane* is the geometry $(\mathbb{H}^2 : \mathcal{M})$ whose points are the points of the open disk

$$\mathbb{H}^2 := \big\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 < 1\big\},$$

and whose transformation group $\mathcal{M}$ is the group generated by reflections in all the circles orthogonal to the boundary circle $\mathbb{A} := \{(x, y) : x^2 + y^2 = 1\}$ of $\mathbb{H}^2$, and by reflections in all the diameters of the circle $\mathbb{A}$. Now $\mathcal{M}$ is indeed a transformation group of $\mathbb{H}^2$: the discussion in 7.1.1 implies that a reflection of the type considered takes points of $\mathbb{H}^2$ to points of $\mathbb{H}^2$ and, being its own inverse, we have the implication $\varphi \in \mathcal{M} \Longrightarrow \varphi^{-1} \in \mathcal{M}$.

We will often call $\mathbb{H}^2$ the *hyperbolic plane*. The boundary circle $\mathbb{A}$ (which is not part of the hyperbolic plane) is called the *absolute*.

7.2.2. We will see later that $\mathcal{M}$ is actually the isometry group of hyperbolic geometry with respect to the *hyperbolic distance*, which will be defined in the next chapter. We will see that although the Euclidean distance between points of $\mathbb{H}^2$ is always less than 2, the hyperbolic plane is unbounded with respect to the hyperbolic distance. Endpoints of a short segment (in the Euclidean sense!) near the absolute are very far away from each other in the sense of hyperbolic distance.

Figure 7.6 gives an idea of what an isometric transformation (the simplest one – a reflection in a line) does to a picture. Note that from our Euclidean point of view, the reflection changes the size and the shape of the picture, whereas from the hyperbolic point of view, the size and shape of the image is exactly the same as that of the original. It should also be clear that *hyperbolic reflections reverse orientation*.

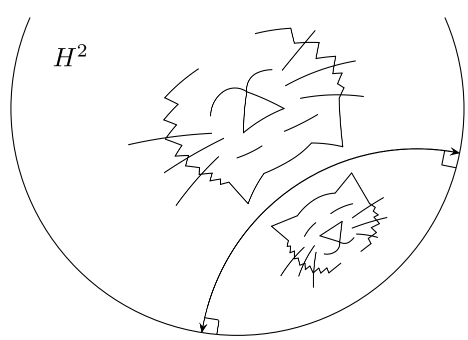

Figure 7.6. An isometry in the hyperbolic plane

7.3. Points and lines in the hyperbolic plane

7.3.1. First we define *points of the hyperbolic plane* simply as points of the open disk $\mathbb{H}^2$. We then define the *lines* on the hyperbolic plane as the intersections with $\mathbb{H}^2$ of the (Euclidean) circles orthogonal to the absolute as well as the diameters (without endpoints) of the absolute (see Fig.7.7).

Note that the endpoints of the arcs and the diameters do not belong to the hyperbolic plane: they lie in the absolute, whose points are not points of our geometry.

Figure 7.7 shows that some lines intersect in one point, others have no common points, and none have two common points (unlike lines in spherical

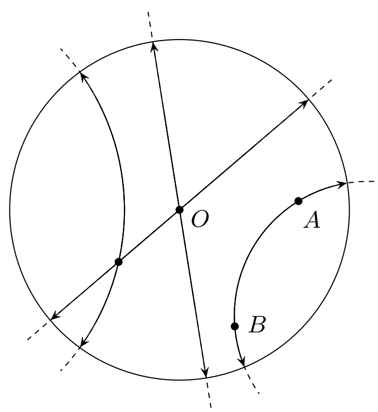

Figure 7.7. Lines on the hyperbolic plane

geometry). This is not surprising, because we have the following statement.

7.3.2. Theorem. *One and only one line passes through any pair of distinct points of the hyperbolic plane.*

Proof. The theorem immediately follows from Corollary 7.1.4, (i). □

7.4. Perpendiculars

7.4.1. Two lines in $\mathbb{H}^2$ are called *perpendicular* if they are orthogonal in the sense of elementary Euclidean geometry. When both are diameters, they are perpendicular in the usual sense, when both are arcs of circles, they have perpendicular tangents at the intersection point, when one is an arc and the other a diameter, then the diameter is perpendicular to the tangent to the arc at the intersection point.

7.4.2. Theorem. *There is one and only one line passing through a given point and perpendicular to a given line.*

Proof. The theorem immediately follows from Corollary 7.1.4, (iv). □

7.5. Parallels and nonintersecting lines

7.5.1. Let l be a line and P be a point of the hyperbolic plane $\mathbb{H}^2$ not contained in the line l. Denote by P and Q the points at which l intersects the absolute. Consider the lines $k = PA$ and $m = PB$ and denote their second intersection points with the absolute by A' and B'. Clearly, the lines k and m do not intersect l. Moreover, any line passing through P between k and m (i.e., any line containing P and joining the arcs AA' and BB') does not intersect l. The lines APA' and BPB' are called *parallels* to l passing through P, and the lines between them are called *nonintersecting lines* with l.

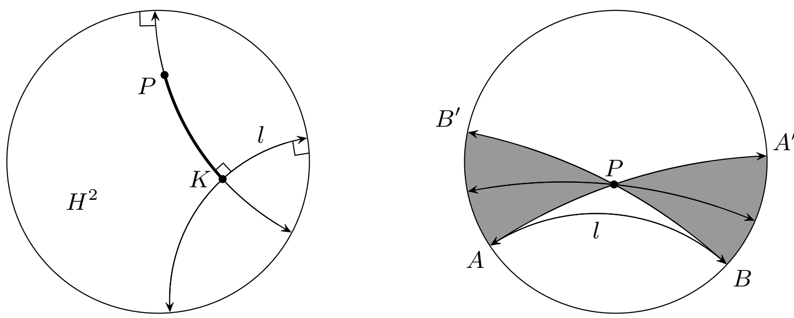

Figure 7.8. Perpendiculars and parallels

We have proved the following statement.

7.5.2. Theorem. *There are infinitely many lines passing through a given point P not intersecting a given line l if $P \notin l$. These lines are all located between the two parallels to l.* □

This theorem contradicts Euclid's famous *Fifth Postulate*, which, in its modern formulation, says that one and only one parallel to a given line passes through a given point. For more than two thousand years, many attempts to prove that the Fifth Postulate follows from Euclid's other postulates (which, unlike the Fifth Postulate, were simple and intuitively obvious) were made by mathematicians and philosophers. Had such a proof been found, Euclidean geometry could have been declared to be an absolute truth both from the physical and the philosophical point of view, it would have been an example of what the German philosopher Kant called the category of synthetic apriori. For two thousand years, the naive belief among scientists in the absolute truth of Euclidean geometry made it difficult for the would be discoverers of other geometries to realize that they had found something worthwhile. Thus the appearance of a consistent geometry in which the Fifth Postulate does not hold was not only a crucial development in the history of mathematics, but one of the turning points in the philosophy of science.

7.6. Sum of the angles of a triangle

7.6.1. Consider three points A, B, C not on one line. The three segments AB, BC, CA (called *sides*) form a *triangle* with *vertices* A, B, C. The *angles* of the triangle, measured in radians, are defined as equal to the (Euclidean measure of the) angles between the tangents to the sides at the vertices.

7.6.2. Theorem. *The sum of the angles α, β, γ of a triangle ABC is less than two right angles:*

$$\boxed{\alpha + \beta + \gamma < \pi}.$$

Proof. In view of Lemma 7.1.3, we can assume without loss of generality that A is O (the center of $\mathbb{H}^2$). But then if we compare the hyperbolic triangle OBC with the Euclidean triangle OBC, we see that they have the same angle at O, but the Euclidean angles at B and C are larger than their hyperbolic counterparts (look at Fig.7.9), which implies the claim of the theorem. $\square$

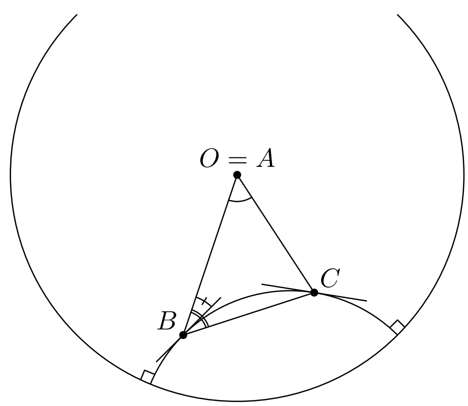

Figure 7.9. Sum of the angles of a hyperbolic triangle

It is easy to see that very small triangles have angles sums very close to π, in fact *the least upper bound of the angle sum of hyperbolic triangles is exactly π*. Further, *the greatest lower bound of these sums is 0.* To see this, divide the absolute into three equal arcs by three points P, Q, R and construct three circles orthogonal to the absolute passing through the pairs of points P and Q, Q and R, R and P. These circles exist by Corollary 7.1.2, item (iii). Then all the angles of the "triangle" PQR are zero, so its angle sum is zero. Of course, PQR is not a real triangle in our geometry (its vertices, being on the absolute, are not points of $\mathbb{H}^2$), but if we take three points P', Q', R'

close enough to P, Q, R, then the angle sum of triangle $P'Q'R'$ will be less than any prescribed $\varepsilon > 0$.

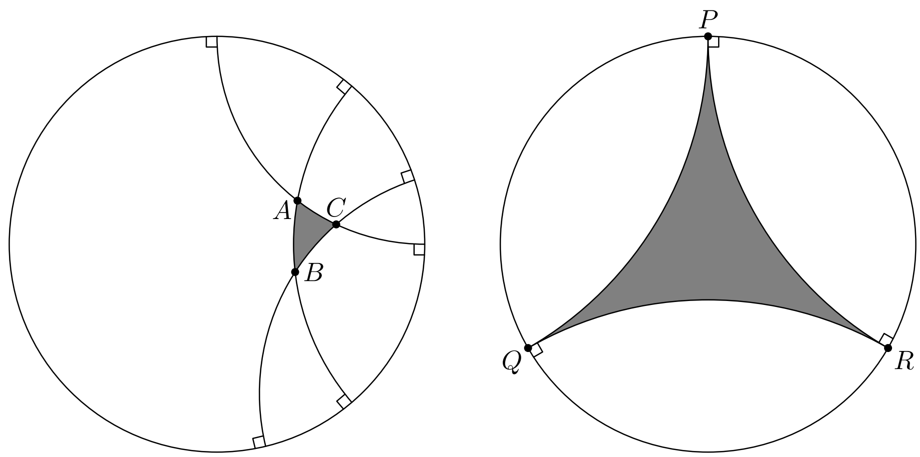

Figure 7.10. Ordinary triangle and "triangle" with angle sum 0

7.7. Rotations and circles in the hyperbolic plane

We mentioned previously that distance between points of the hyperbolic plane will be defined later. Recall that the hyperbolic plane is the geometry $(\mathbb{H}^2 : \mathcal{M})$, in which, by definition, $\mathcal{M}$ is the transformation group generated by all reflections in all the lines of $\mathbb{H}^2$. If we take the composition of two reflections in two intersecting lines, then what we get should be a "rotation", but we can't assert that at this point, because we don't have any definition of rotation: the usual (Euclidean) definition of a rotation or even that of a circle cannot be given until distance is defined.

But the notions of rotation and of circle *can* be defined without appealing to distance in the following natural way: a *rotation* about a point $P \in \mathbb{H}^2$ is, by definition, the composition of any two reflections in lines passing through P. If O and A are distinct points of $\mathbb{H}^2$, then the (hyperbolic) *circle* of center O and radius OA is the set of images of A under all rotations about O.

7.7.1. Theorem. *A (hyperbolic) circle in the Poincaré disk model is a Euclidean circle, and vice versa, any Euclidean circle inside $\mathbb{H}^2$ is a hyperbolic circle in the geometry $(\mathbb{H}^2 : \mathcal{M})$.*

Proof. Let $\mathcal{C}$ be a circle of center I and radius OA in the geometry $(\mathbb{H}^2 : \mathcal{M})$. Using Lemma 7.1.3, we can send I to the center O of $\mathbb{H}^2$ by a reflection φ. Let ρ be a rotation about I determined by two lines l_1 and

l_2. Then the lines $d_1 := \varphi(l_1)$ and $d_2 := \varphi(l_2)$ are diameters of the absolute and the composition of reflections in these diameters is a Euclidean rotation about O (and simultaneously a hyperbolic one). This rotation takes the point $\varphi(A)$ to a point on the circle $\mathcal{C}'$ of center O and radius $O\varphi(A)$, which is simultaneously a hyperbolic and Euclidean circle. Now by Corollary 7.1.4 item (i), the inverse image of $\varphi^{-1}(\mathcal{C}')$ will be a (Euclidean!) circle. But $\varphi^{-1}(\mathcal{C}')$ coincides with $\mathcal{C}$) by construction, so $\mathcal{C}$) is indeed a Euclidean circle in our model.

The proof of the converse assertion is similar and is left to the reader (see Exercise 7.7).

7.8. Hyperbolic geometry and the physical world

In his famous book *Science et Hypothèse*, Henri Poincaré describes the physics of a small "universe" and the physical theories that its inhabitants would create. The universe considered by Poincaré is Euclidean, plane (two-dimensional), has the form of an open unit disk. Its temperature is 100° Farenheit at the center of the disk and decreases linearly to absolute zero at its boundary. The lengths of objects (including living creatures) are proportional to temperature.

How will a little flat creature endowed with reason and living in this disk describe the main physical laws of his universe? The first question he/she may ask could be: Is the world bounded or infinite? To answer this question, an expedition is organized; but as the expedition moves towards the boundary of the disk, the legs of the explorers become smaller, their steps shorter – they will never reach the boundary, and conclude that the world is infinite.

The next question may be: Does the temperature in the universe vary? Having constructed a thermometer (based on different expansion coefficients of various materials), scientists carry it around the universe and take measurements. However, since the lengths of all objects change similarly with temperature, the thermometer gives the same measurement all over the universe – the scientists conclude that the temperature is constant.

Then the scientists might study straight lines, i.e., investigate what is the shortest path between two points. They will discover that the shortest path is what we perceive to be the arc of the circle containing the two points and orthogonal to the boundary disk (this is because such a circular path brings the investigator nearer to the center of the disk, and thus increases the length of his steps). Further, they will find that the shortest path is unique and regard such paths as "straight lines".

Continuing to develop geometry, the inhabitants of Poincaré's little flat universe will decide that there is more than one parallel to a given line passing through a given point, the sum of angles of triangles is less than π, and obtain other statements of hyperbolic geometry.

Thus they will come to the conclusion that they live in an infinite flat universe with constant temperature governed by the laws of hyperbolic geometry. But this not true – their universe is a finite disk, its temperature is variable (tends to zero towards the boundary) and the underlying geometry is Euclidean, not hyperbolic!

The philosophical conclusion of Poincaré's argument is not agnosticism – he goes further. The physical model described above, according to Poincaré, shows not only that the truth about the universe cannot be discovered, but that it makes no sense to speak of any "truth" or approximation of truth in science – pragmatically, the inhabitants of his physical model are perfectly right to use hyperbolic geometry as the foundation of their physics because it is convenient, and it is counterproductive to search for any abstract Truth which has no practical meaning anyway.

This conclusion has been challenged by other thinkers, but we will not get involved in this philosophical discussion.

7.8. Problems

7.1. Prove that inversion maps circles and straight lines to circles or straight lines.

7.2. Prove that inversion is conformal (i.e., it preserves the measure of angles).

7.3. Prove that inversion maps any circle orthogonal to the circle of inversion into itself.

7.4. Prove that if P is point lying outside a circle γ and A, B are the intersection points with the circle of a line l passing through P, then the product $|PA| \cdot |PB|$ (often called the *power of P with respect to γ*) does not depend on the choice of l.

7.5. Prove that if P is point lying inside a circle γ and A, B are the intersection points with the circle of a line l passing through P, then the product $|PA| \cdot |PB|$ (often called the *power of P with respect to γ*) does not depend on the choice of l.

7.6. Prove that inversion with respect to a circle orthogonal to a given circle $\mathcal{C}$ maps the disk bounded by $\mathcal{C}$ bijectively onto itself.

7.7. Prove that any Euclidean circle inside the disk model is also a hyperbolic circle. Does the ordinary (Euclidean) center coincide with its "hyperbolic center"?

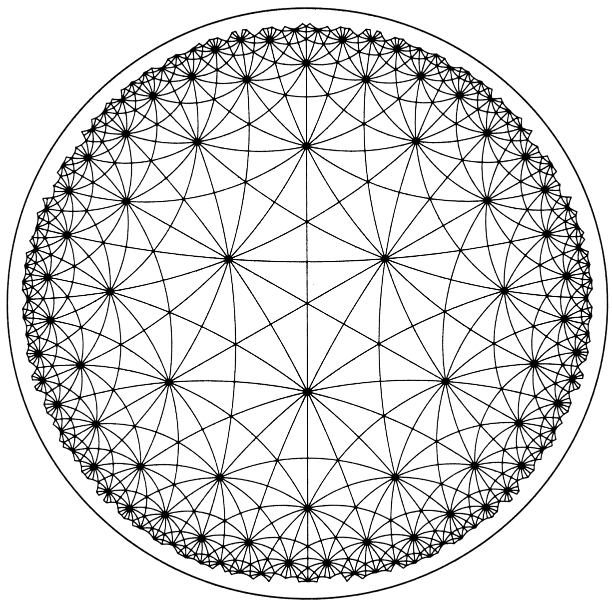

Figure 7.11. A pattern of lines in $\mathbb{H}^2$

7.8. Study Figure 7.11. Does it demonstrate any tilings of $\mathbb{H}^2$ by regular polygons? Of how many sides? Do you discern a Coxeter geometry in this picture with "hyperbolic Coxeter triangles" as fundamental domains? What are their angles?

7.9. Prove that any inversion of $\overline{\mathbb{C}}$ preserves the cross ratio of four points:

$$\langle z_1, z_2, z_3, z_4 \rangle := \frac{z_3 - z_1}{z_3 - z_2} : \frac{z_4 - z_1}{z_4 - z_2}.$$

7.10*. Using complex numbers, invent a formula for the distance between points on the Poincaré disk model and prove that "symetry with respect to straight lines" (i.e., inversion) preserves this distance.

7.11. Prove that hyperbolic geometry is homogeneous in the sense that for any two flags (i.e., half planes with a marked point on the boundary) there exists an isometry taking one flag to the other.

7.12. Prove that the hyperbolic plane (as defined via the Poincaré disk model) can be tiled by regular pentagons.

7.13. Define inversion (together with the center and the sphere of inversion) in Euclidean space $\mathbb{R}^3$, state and prove its main properties: inversion takes planes and spheres to planes or spheres, any sphere orthogonal to the sphere of inversion to itself, any plane passing through the center of inversion to itself.

7.14. Using the previous exercise, prove that any inversion in $\mathbb{R}^3$ takes circles and straight lines to circles or straight lines.

7.15. Prove that any inversion in $\mathbb{R}^3$ is conformal (preserves the measure of angles).

7.16. Construct a model of hyperbolic space geometry on the open unit ball (use Exercise 7.13).