

Lecture 13 GALOIS GEOMETRIES

§1. Finite fields

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Theorem $\forall$ prime p $\forall n \in \mathbb{N}$ $\exists!$ field $\mathbb{F}_q$ with $q=p^n$ elements; conversly, if $\mathbb{F}$ is a finite field, then $\exists!$ $q=p^n$ s.t. $\mathbb{F}=\mathbb{F}_q$.

Example: $p=2, n=2$. $\mathbb{F}_4=\{0,1,2,3\}$; the addition table is shown above, the multiplication table is that of $\mathbb{Z}/3\mathbb{Z}$.

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§2. Examples of finite geometries

(1.) Aff_5^2: affine plane of order 5. It has 25 points, Fig. 1
30 lines, 5 points on each line, 6 lines passing through each point
The lines are constructed as in Euclidean geometry, i.e. for fixed (x_0,y_0) and $\{X,Y\}$, $\ell=\{x=x_0+Xt, y=y_0+Yt \mid t\in\mathbb{F}_5\}$
The transformation group for Aff_5^2 consists of all bijections of Aff_5^2 that take lines to lines.

(2) The Fano projective plane: 7 points, 7 lines, 3 points on each line, 3 lines through each point. Duality! Fig. 2
The transformation group consists of line-preserving bijections

(3) Aff_4^2: affine plane of order 4. It has 16 points, Fig 3
20 lines, 4 points on each line, 5 lines through each point.
The transformation group consists of all line-preserving bijections

(4) The projective plane of order 5 ($\mathbb{F}_4P^2$) It has 21 points, Fig. 4
21 lines, 5 points on each line, 5 lines through each pt.
The transformation group consists of all line-preserving bijections.
This geometry has the following properties:

I. $\forall P,Q\ \exists!\ \ell \ni P\cup Q$ I*. $\forall \ell_1,\ell_2\ \exists!\ P\in \ell_1\cap\ell_2$

II. $\exists$ 4 points that determine six distinct lines

II* $\exists$ 4 lines that determine six distinct points

Note that Aff_4^2 is a subgeometry of the projective plane $\mathbb{F}_4P^2$

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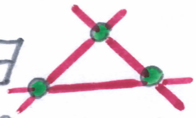
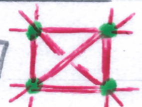
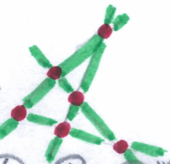
§3. Axioms for finite affine geometries

A1. $\exists!$  A2. $\exists!$  A3. $\exists$ 

Theorem (1) $\forall$ prime p, $\forall m \in \mathbb{N}$ $\exists$ geometry $Aff^2_{q=p^m}$ (with q^2 points, q^2+q lines, q points on each line, $q+1$ lines through each point) which satisfies A1–A3 and its transf. group is $G=\{$line-preserving bijections$\}$.

(2) For any geometry A satisfying A1–A3, $\exists q=p^m$ (p prime) s.t. $A = Aff^2_q$

§4. Axioms for finite projective geometries

$\mathcal{P}1.\ \exists!$  ; $\mathcal{P}1^*\ \exists!$  ; $\mathcal{P}_2\ \exists$  ; $\mathcal{P}_2^*\ \exists$ 

Theorem (1) Any finite geometry Π satisfying axioms $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_1^*, \mathcal{P}_2^*$ has the same number of points and lines (denoted by $n+1$); the number n is called the order of Π.

(2) If $n=p^m$, where p is prime, $m \in \mathbb{N}$, then a projective geometry of order n exists

Remarks 1. The order n does not determine the geometry: For example, there are several projective geometries of order 9.

2. It is not known for what values of n there are projective geometries of order n. There are none for $n=10$ (supercomputer proof in 1991). For $n=12$ this is an open question.

3. In the case $n=p^m$ (p-prime, $m \in \mathbb{N}$), different affine geometries are obtained from a projective geometry of order n by throwing out one projective line, depending on the choice of the line

§5 Open problems

- For what n are there projective geometries of order n?

Conjecture 1. The order of any finite projective plane is of the form $n=p^m$, where p is prime and $m \in \mathbb{N}$.

Conjecture 2. The Desargues theorem holds in any finite projective plane of prime order $n=p$.