Lecture 2. THE PLATONIC SOLIDS

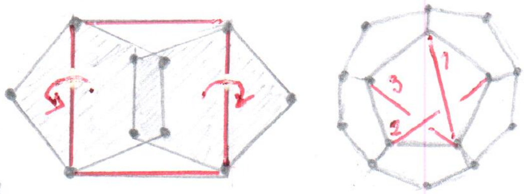
§1. Regular polyhedra

Рис. 66

Definitions Convex polyhedron $P = \mathrm{Conv}(\because\because)$; face, edge, vertex; regular polyhedron := convex polyhedron inscribed in S^2, faces are reg. polygons, links of vertices are reg. polygons.

Examples (1) tetrahedron $F=4, E=6, V=4$; $\langle 3,3\rangle$; $|G|=24$
(2) cube $(6,12,8)$, $\langle 4,3\rangle$, 48. (3) octahedron $(8,12,6)$; $\langle 3,4\rangle$, 48
(4) dodecahedron $(12,30,20)$, $\langle 5,3\rangle$, 120 (5) icosahedron $(20,30,12)$; $\langle 3,5\rangle$, 12

Duality Рис.3.4
∃ dodecahedron
Five Kepler cubes



1,2,3,4,5

§2. Classification

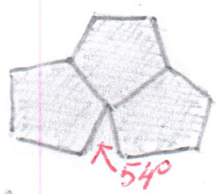
Theorem 1. {regular polyhedra} ≡ {△ ◻ ◇ ⬠ ⬡}

Proof faces ∈ {△, ◻, ⬠} ⇐ 3,4,5 edges at each vertex

Def.: defect $(v) := 360° - \sum(\text{angles at } v)$

(i) ◻ : defect = 90° ⇒ cube

(ii) ⬠ : defect = 54° ⇒ dodecahedron



(iii) △ : defect = 180° ⇒ tetrahedron

(iv) △ : defect = 120° ⇒ octahedron

(v) △ : defect = 60° ⇒ icosahedron □

Remark There is a topological proof (Euler characteristic) and an algebraic proof (finite subgroups of $SO(3) = \mathrm{Sym}\, S^2$) see the book

§3. Platonic solids in philosophy, art, science

Plato: (earth ◻, fire △, air ◇, water ⬡) + stars etc. dodecaэдр
da Vinci Рис.3.1, Kepler Рис.3.2

§4. Higher dimensions

Inductive definition: $RP_i^d :=$ convex d-dimensional polyhedron inscribed in S^{d-1}, faces are RP_j^{d-1}, links are RP_k^{d-1}.

Theorem 2 There are six 4-dimensional regular polyhedra $\boxed{\text{Рис. 3.7}}$

Theorem 3. For $d \geq 5$, there are three d-dimensional reg. polyhedra: the regular simplex, the hypercube, the cocube. $\boxed{\text{Рис. 3.8}}$

Schläfli symbol of RP^d $(r_1, r_2, \dots r_{d-2}, r_{d-1})$ means that RP^3 has r_{d-1} faces at each face of dimension $d-3$ with Schläfli symbol $(r_1, \dots, r_{d-1})$

Examples: square $\langle 4 \rangle$, cube $\langle 4,3 \rangle$, dodecahedron $\langle 5,3 \rangle$

$d=4$: $\langle 3,3,3 \rangle, \langle 4,3,3 \rangle, \langle 3,3,4 \rangle, \langle 3,4,3 \rangle \langle 5,3,3 \rangle \langle 3,3,5 \rangle$

$d \geq 5$: $\langle 3,3,\dots,3,3 \rangle, \langle 4,3,\dots,3,3 \rangle, \langle 3,3,\dots,3,4 \rangle$ simplex, hypercube, cocube

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§5 Fundamental domains of Platonic solids

Definition $(X : G)$ Klein geometry, $X \subseteq \mathbb{R}^{n=3}$, $F \subset X$ is a fundamental domain if:
- $F \subset X$ open
- $F \cap Fg = \emptyset \quad \forall g \neq id$
- $\bigcup_{g \in G} \overline{Fg} = X$

For the Platonic solids, the fundamental domains are: $\boxed{\text{Рис. 3.5}}$
