On quivers and their representations[1]

By definition, a quiver Γ is an oriented graph. It consists of a set Γ_0 of vertices and a set Γ_1 of edges (also called arrows). For any arrow $a \in \Gamma_1$ we denote by $s(a)$ its *source* and by $t(a)$ its *target*. Today we will consider only finite quivers, that is, the sets Γ_0 and Γ_1 are finite.

A representation of a quiver Γ over a field k is by definition a collection $(M_i, i \in \Gamma_0; m_a, a \in \Gamma_1)$, where M_i are vector spaces and $m_a \colon M_{s(a)} \to M_{t(a)}$ are linear maps. A representation is finite-dimensional if all M_i are finite-dimensional. A morphism of representations $(M_i, m_a) \to (N_i, n_a)$ is a collection $(f_i, i \in \Gamma_0)$, where $f_i \colon M_i \to N_i$ are linear maps obeying equalities

$$f_{t(a)} m_a = n_a f_{s(a)}$$

for any $a \in \Gamma_1$. Representations of Γ over k form an abelian category which we denote by $Rep_{\mathsf{k}}\Gamma$.

Also one can consider contravariant representations of Γ, where linear maps go in the opposite direction. If we denote the quiver Γ with all the arrows reversed by Γ^{op}, then the category of contravariant representations of Γ is just $Rep_{\mathsf{k}}\Gamma^{op}$.

A *path* from $i_0 \in \Gamma_0$ to $i_n \in \Gamma_0$ is by definition a sequence of arrows $a_1, \ldots, a_n$ such that $s(a_1) = i_0$, $t(a_n) = i_n$ and $t(a_k) = s(a_{k+1})$ for all $k = 1, \ldots, n-1$. Such path is written as $p = a_n a_{n-1} \cdot \ldots \cdot a_1$, it is said to have source $s(p) = i_0$, target $t(p) = i_n$ and length n. Clearly, any arrow is a path of length 1. Also, by definition for any $i \in \Gamma_0$ we have a path of length 0 from i to i, denoted by e_i.

The path algebra $\mathsf{k}\Gamma$ of Γ is defined as follows. As a k-vector space, it has the basis formed by all paths in Γ. The composition law is defined on basic elements $p = a_n \ldots a_1$ and $q = b_m \ldots b_1$ by $pq = a_n \ldots a_1 b_m \ldots b_1$ if $t(q) = s(p)$ and $pq = 0$ otherwise. The algebra $\mathsf{k}\Gamma$ is associative, it has the identity $1 = \sum_{i \in \Gamma_0} e_i$. The path algebra is finite-dimensional if and only if Γ has no oriented cycles. The algebra $\mathsf{k}\Gamma$ is usually non-commutative. Also, $\mathsf{k}\Gamma$ has a grading by the path length: $\mathsf{k}\Gamma = \oplus_{n \geqslant 0} (\mathsf{k}\Gamma)_n$.

Let $R = R(\mathsf{k}\Gamma) = \oplus_{n>0} (\mathsf{k}\Gamma)_n \subset \mathsf{k}\Gamma$ be the subspace spanned by paths of positive length. Clearly, R is a two-sided ideal. One has an algebra isomorphism $\mathsf{k}\Gamma / R(\mathsf{k}\Gamma) \cong \prod_{i \in \Gamma_0} \mathsf{k}$, hence $\mathsf{k}\Gamma$ is a basic algebra.

Elements $e_i, i \in \Gamma_0$ form a complete family of orthogonal idempotents: it means that

$$e_i^2 = e_i, e_i e_j = 0 \quad \text{for } i \neq j \text{ and } \quad \sum_i e_i = 1. \tag{1}$$

Instead of representations one can consider modules over path algebras. For a right $\mathsf{k}\Gamma$-module M and $i \in \Gamma_0$ denote

$$M_i := M \cdot e_i \subset M.$$

Formulas (1) imply the equality of vector spaces $M = \oplus_{i \in \Gamma_0} M_i$. For any arrow $i \xrightarrow{a} j$ the right multiplication $M \to M$ sends M_j to M_i and other M_k-s to 0. Thus we get a contravariant representation of Γ. Vice versa, for any contravariant representation (M_i, m_a) of Γ consider the vector space $M = \oplus_i M_i$. It is a right $\mathsf{k}\Gamma$-module: let idempotents e_i acts by projectors to M_i, let an arrow $a \colon i \to j$ act by m_a on M_j and by 0 on other M_k-s. This action extends to a right $\mathsf{k}\Gamma$ action on M. This gives a proof of

Proposition 1. *One has equivalences*

$$Rep_{\mathsf{k}}\Gamma \cong \mathsf{k}\Gamma{-}\mathrm{Mod}; Rep_{\mathsf{k}}\Gamma^{op} \cong \mathrm{Mod}{-}\mathsf{k}\Gamma.$$

[1]By technical reasons these notes are typed in English.

We prefer to use right modules/contravariant representations. By default, modules are right modules.

Let us consider the following $\mathsf{k}\Gamma$-modules.

For any $i \in \Gamma_0$ let S_i be the representation of Γ such that $(S_i)_i = \mathsf{k}$, $(S_i)_j = 0$ otherwise, all arrows act by zero. Let $P_i = e_i\mathsf{k}\Gamma \subset \mathsf{k}\Gamma$ be the cyclic submodule generated by e_i.

Proposition 2. *1. Modules S_i are simple and pairwise non-isomorphic.*

2. One has $\mathsf{k}\Gamma \cong \oplus_{i\in\Gamma_0} P_i$. Modules P_i are graded, projective, indecomposable and pairwise non-isomorphic.

3. For any right $\mathsf{k}\Gamma$-module M one has $\mathrm{Hom}(P_i, M) \cong M_i$. In particular, $\mathrm{Hom}(P_i, S_j) = \delta_{ij}\mathsf{k}$.

4. $\mathrm{Hom}(P_i, P_j) = (P_j)_i = \langle$*paths from i to j*$\rangle_{\mathsf{k}}$.

Proof. (1) is clear.

(2) Identities (1) imply that $\mathsf{k}\Gamma \cong \oplus_{i\in\Gamma_0} P_i$, hence P_i-s are projective. Any P_i is graded as a submodule in $\mathsf{k}\Gamma$ generated by a homogeneous element e_i. One has $P_i/P_iR(\mathsf{k}\Gamma) \cong S_i$ hence P_i-s are pairwise non-isomorphic.

(3) To any homomorphism $f\colon P_i \to M$ one can associate an element $f(e_i) \in M_i$. This element defines f uniquely. Moreover, any $m \in M_i$ can be such an image for some f. Indeed, consider the homomorphism $\bar{f}\colon \mathsf{k}\Gamma \to M$ given by $\bar{f}(x) = mx$ and restrict it to $P_i \subset \mathsf{k}\Gamma$, denote the restriction by f. Then $f(e_i) = me_i = m$.

(4) follows from (3) because $(P_j)_i = e_j\mathsf{k}\Gamma e_i = \langle$paths from i to $j\rangle_{\mathsf{k}}$. □

By a *relation* in Γ we mean an element in $\mathsf{k}\Gamma$ of the form $\sum_{i=1}^n \lambda_i p_i$ where $\lambda_i \in \mathsf{k}$ and $p_1, \ldots, p_n$ are paths with common source and target. Alternatively, we can consider a two-sided ideal $I \subset k\Gamma$. Since $I = \oplus_{i,j\in\Gamma_0} e_iIe_j$, any ideal is generated (over k) by relations.

For an ideal $I \subset \mathsf{k}\Gamma$ a *path algebra with relations* is defined as $\mathsf{k}\Gamma/I$. Today we will denote this algebra by A. We usually use the same notations for elements in $\mathsf{k}\Gamma$ and their images in A. In particular, we have a complete system of orthogonal idempotents $e_i, i \in \Gamma_0$ in A.

For any $i, j \in \Gamma_0$ we denote $(\mathsf{k}\Gamma)_{ij} := e_i\mathsf{k}\Gamma e_j \subset \mathsf{k}\Gamma$, $I_{ij} := e_iIe_j \subset I$ and $A_{ij} = e_iAe_j \subset A$, these are linear subspaces. We have $\mathsf{k}\Gamma = \oplus_{ij}(\mathsf{k}\Gamma)_{ij}$, $I = \oplus_{ij} I_{ij}$ and $A = \oplus_{i,j} A_{ij}$. Consequently,

$$A_{ij} = (k\Gamma)_{ij}/I_{ij},$$

it is the space of paths from j to i modulo some relations.

As we shall see, one can restrict to admissible ideals in path algebras.

Definition 3. An ideal $I \subset \mathsf{k}\Gamma$ is called *admissible* if $R(\mathsf{k}\Gamma)^2 \supset I \supset R(\mathsf{k}\Gamma)^N$ for some N.

For an admissible ideal the path algebra A with relations is finite-dimensional. Note also that zero ideal in $\mathsf{k}\Gamma$ is admissible iff Γ has no oriented cycles.

Further we will assume that ideal I is admissible.

Let $R(A) := R(\mathsf{k}\Gamma)/I \subset A$, this is a nilpotent ideal. Moreover, $A/R(A) \cong \mathsf{k}\Gamma/R(\mathsf{k}\Gamma) \cong \prod_{i\in\Gamma_0} k$, hence $R(A)$ is the radical of A. Also it follows that A is a basic algebra.

In view of Proposition 1, modules over A correspond to those representations of Γ that obey relations from I. This gives an equivalence of certain categories.

Let M be a module over A. For any $i \in \Gamma_0$ we denote by M_i the subspace $Me_i \subset M$. As above, one has a decomposition of k-vector spaces

$$M = \oplus_{i\in\Gamma_0} M_i.$$

Simple $\mathsf{k}\Gamma$-modules S_i satisfy all relations from I hence S_i is an A-module. Also, let $P_i := e_iA \subset A$, this is a right A-module.

Proposition 4. *Assume $I \subset \mathsf{k}\Gamma$ is an admissible ideal and $A = \mathsf{k}\Gamma/I$. Then*

1. *Modules S_i are simple and pairwise non-isomorphic. Any simple A-module is one of these.*
2. *One has $A \cong \oplus_{i\in\Gamma_0} P_i$. Modules P_i are projective, indecomposable and pairwise non-isomorphic. Any indecomposable projective A-module is one of these.*
3. *For any right A-module M one has $\mathrm{Hom}(P_i, M) \cong M_i$. In particular, $\mathrm{Hom}(P_i, S_j) = \delta_{ij}\mathsf{k}$.*
4. $\mathrm{Hom}(P_i, P_j) = (P_j)_i = \langle$*paths from i to j*$\rangle_{\mathsf{k}}/I_{ji}$.

Proof. (1) Only the last assertion in not straightforward. It follows from a more general fact: any finite-dimensional A-module has a filtration with quotients S_i. Indeed, let M be a such module. Consider the sequence $M \supset M \cdot R(A) \supset M \cdot R(A)^2 \supset \ldots \supset M \cdot R(A)^N = 0$. Any its quotient is a module annihilated by $R(A)$, thus is a direct sum of S_i-s. Refining this filtration we get a filtrations with quotients S_i-s.

The proof of (2) is the same as in Proposition 2. To see that P_i is indecomposable, use that $P_i/P_i \cdot R(A) \cong S_i$ is indecomposable and Lemma 5 below. To see that any indecomposable projective A-module is isomorphic to some P_i, use Krull-Schmidt theorem.

(3),(4) are similar to those of Proposition 2. □

Lemma 5 ("Nakayama's Lemma")**.** *Assume $I \subset \mathsf{k}\Gamma$ is an admissible ideal and $A = \mathsf{k}\Gamma/I$. Let M be an A-module such that $M = M \cdot R(A)$. Then $M = 0$.*

Proof. Clear since $R(A)^N = 0$ for some N. □

Corollary 6. *Assume $I \subset \mathsf{k}\Gamma$ is an admissible ideal and $A = \mathsf{k}\Gamma/I$. Then any finite-dimensional A-module has a finite filtration with quotients S_i, $i \in \Gamma_0$.*

Now let us prove that representation theory of finite-dimensional algebras reduces to the study of path algebras with admissible relations (over an algebraically closed field).

Let us denote by proj$-A$ the category of finite-dimensional projective right A-modules. Then we have

Proposition 7. *Let $A = \mathsf{k}\Gamma/I$ be the path algebra with admissible relations. Then the Auslander-Reiten quiver $\Gamma(\mathrm{proj}-A)$ of the category* proj$-A$ *is isomorphic to* Γ.

Proof. Recall the definition of the Auslander-Reiten quiver of a k-linear Krull-Schmidt category. Its vertices are isomorphism classes of indecomposable objects, number of arrows from X to Y is the dimension of the space

$$\mathrm{Irr}(X, Y) := \mathcal{R}(X, Y)/\mathcal{R}^2(X, Y),$$

where $\mathcal{R}$ denotes the radical of the given category.

In our case, indecomposable objects in proj$-A$ are exactly the modules P_i, $i \in \Gamma_0$. Further, we have by Proposition 4

$$\mathcal{R}(P_i, P_j) = \langle\text{paths of length} \geqslant 1 \text{ from } i \text{ to } j\rangle_{\mathsf{k}}/I_{ji}$$

and

$$\mathcal{R}^2(P_i, P_j) = \langle\text{paths of length} \geqslant 2 \text{ from } i \text{ to } j\rangle_{\mathsf{k}}/I_{ji},$$

hence

$$\mathcal{R}(P_i, P_j)/\mathcal{R}^2(P_i, P_j) \cong$$
$$\cong \langle \text{paths of length} \geqslant 1 \text{ from } i \text{ to } j\rangle_{\mathsf{k}}/\langle \text{paths of length} \geqslant 2 \text{ from } i \text{ to } j\rangle_{\mathsf{k}} \cong$$
$$\cong \langle \text{paths of length 1 from } i \text{ to } j\rangle_{\mathsf{k}} = \langle \text{arrows from } i \text{ to } j\rangle_{\mathsf{k}}.$$

Thus the dimension of $\mathrm{Irr}(P_i, P_j)$ equals to the number of arrows from i to j in Γ. □

Definition 8. A finite-dimensional k-algebra A is called *elementary* if $A/R(A) \cong \prod \mathsf{k}$.

From lecture 3 it follows that A is elementary iff the right A-module A decomposes as $A = \oplus_{i=1}^n P_i$ where $P_1, \ldots, P_n$ are indecomposable pairwise non-isomorphic modules and for all i one has $(\mathrm{End}\, P_i)/R(\mathrm{End}\, P_i) \cong \mathsf{k}$.

Note that any elementary algebra is basic. Also we have

Proposition 9. *Suppose* k *is algebraically closed. Then a finite-dimensional algebra* A *is basic if and only if it is elementary.*

Proof. Recall that an algebra A is basic iff the right A-module A decomposes as $A = \oplus_{i=1}^n P_i$ where $P_1, \ldots, P_n$ are indecomposable pairwise non-isomorphic modules. Hence it suffices to check that any division algebra $T(P_i) := (\mathrm{End}\, P_i)/R(\mathrm{End}\, P_i)$ is just k. But $T(P_i)$ is a finite-dimensional division k-algebra, it must be k since k is algebraically closed. □

Theorem 10 (Gabriel)**.** *Let* A *be an elementary finite-dimensional* k*-algebra. Then* A *is isomorphic to* $\mathsf{k}\Gamma/I$ *where* Γ *is a finite quiver and* I *is an admissible ideal. Moreover, such quiver* Γ *is unique.*

Proof. Uniqueness of Γ follows from Proposition 7. Let us prove the first assertion.

Let $A \cong \oplus_{i=1}^n P_i$ be the decomposition into indecomposable projective modules, then

$$A \cong \mathrm{End}_{\mathrm{mod}-A} A \cong \oplus_{i,j} \mathrm{Hom}(P_i, P_j).$$

Since A is elementary, for any i one has $\mathrm{End}\, P_i = \mathsf{k} \oplus R(\mathrm{End}\, P_i)$. Denote by $e_i \in A$ the element $1_{P_i} \in \mathrm{End}\, P_i \subset A$, then $e_1, \ldots, e_n \in A$ are orthogonal idempotents.

Let $\Gamma_0 := \{1, \ldots, n\}$. For any i, j take $\dim_{\mathsf{k}}(e_j(R(A)/R(A)^2)e_i)$ arrows from i to j. The finite quiver Γ is ready!

Choose bases in the vector spaces $e_j(R(A)/R(A)^2)e_i$, choose their lifts to $e_jR(A)e_i$. Define a homomorphism $f\colon \mathsf{k}\Gamma \to A$ by sending $e_i \in \mathsf{k}\Gamma$ to $e_i \in A$ and any arrow from i to j in Γ to the corresponding element in $e_jR(A)e_i$.

Let us check that f is surjective. We see that f sends $R(\mathsf{k}\Gamma)$ to $R(A)$ and thus $R(\mathsf{k}\Gamma)^m$ to $R(A)^m$ for any $m \geqslant 1$. The map

$$f_m\colon R(\mathsf{k}\Gamma)^m/R(\mathsf{k}\Gamma)^{m+1} \to R(A)^m/R(A)^{m+1}$$

induced by f is a bijection for $m = 0$ (clear) and $m = 1$ (by the construction of Γ), and thus f_m is epimorphic for any $m \geqslant 0$. Consequently, f induces surjective maps $\mathsf{k}\Gamma/R(\mathsf{k}\Gamma)^m \to A/R(A)^m$ for any $m \geqslant 1$, which are isomorphisms for $m = 1, 2$. Since $R(A)$ is a radical, one has $R(A)^N = 0$ for some N. It follows that f is surjective.

Let $I = \ker f$, then $A \cong \mathsf{k}\Gamma/I$. It remains to check that I is admissible. First, $f(R(\mathsf{k}\Gamma)^N) \subset R(A)^N = 0$ by the above, hence $R(\mathsf{k}\Gamma)^N \subset I$. Also, the map $\mathsf{k}\Gamma/R(\mathsf{k}\Gamma)^2 \to A/R(A)^2$ induced by f is an isomorphism, hence $I \subset f^{-1}(R(A)^2) = R(\mathsf{k}\Gamma)^2$. □