Lecture 8

VECTOR FIELDS ON SURFACES

In this lecture, we discuss vector fields on orientable surfaces. Here we will see that there is a deep relationship between the *global* topological properties of the surface and the structure of vector fields on it, namely the (*local*!) characteristics of its singular points. The previous lecture will serve as the local version of the theory.

8.1. What is a Vector Field on a Surface?

A simple example of a vector field on a surface is the velocity field of points on the 2-sphere rotating with constant speed around the N-S axis. In order to define this notion mathematically in the general case, we will assume that our (compact closed orientable) surface M is embedded in $\mathbb{R}^3$. This means that M can be covered by a finite number of open disks $\{U_k\}$ each of which is the graph (график in Russian) of a univalent function $z_k = F_k(x_k, y_k)$ with respect to an orthonormal system of coordinates (O_k, x_k, y_k, z_k) (called *local coordinates*).

Thus *locally* the situation here is the same as in the previous lecture: one can define the smooth vector fields, trajectories, singular points of a vector field, generic vector fields, the index of a vector field at a singular point, etc. However, *for an arbitrary curve* $\gamma\colon \mathbb{S}^1 \to M$, *the index of a vector field* Ind(V, γ) *cannot be correctly defined*, because the Gauss map uses the parallel shift of vectors to a common origin, and such a shift is not well defined on the whole surface. Nevertheless, Theorems 7.1 and 7.2 of the previous lecture remain valid *provided* that they are understood locally, i.e., as taking place in a disk $U_k \subset M$.

Remark. A more appropriate setting for this lecture is the framework of smooth surfaces (2-dimensional differentiable manifolds), where the vector field consists of vectors lying in the so-called "tangent planes" to the surface. Since this notion is not known to the listeners of this course, we have to resort the elementary approach given above, which involves no tangent planes.

The *index of a generic vector field* V on a closed compact orientable surface M is the sum of all indices for all singular points of this field (we denote it by $\mathrm{Ind}(M, V)$).

As for the case of a plane, a *generic vector field on a surface* M is defined as a generic vector field on all the U_k with a finite number of singular points, all of which are generic (i.e., are nodes, or foci, or saddles).

8.2. Two Lemmas

The two following lemmas will be needed in the proof of the main result of this lecture, the Poincaré Theorem.

Lemma 8.1. *If p is a nonsingular point of a generic vector field V, D is a disk centered at p, and V_0 is any nonzero vector, then there exists another vector field W with the same singular points, coinciding with V outside of D and such that $W(p) = V_0$.*

Proof. By continuity, there is a disk D_0 concentric to D such that all the vectors $V(q)$, $q \in D_0$, have a direction that differs by less than $1°$ from the direction of $V(p)$. Let r be the radius of D_0, α be the angle between $V(p)$ and V_0, and S^1_s be the circle of radius $s \le r$ centered at p. Then the required vector field W is obtained from V by rotating all the vectors $V(m)$, $m \in S^1_s$, by the angle $\alpha(r-s)/r$ and replacing $V(p)$ by V_0.

Lemma 8.2. *For any generic vector field V on a surface M, there is a triangulation of M such that any open 2-simplex contains no more than one singular point.*

Proof. Since the number of singular points is finite, by slightly moving the vertices of the triangulation, we can ensure that no singular point is a vertex or a point of an edge of the triangulation. By performing iterated barycentric subdivisions a sufficient number of times, we can ensure that there is no more than one singular point in each closed 2-simplex. Then we again slightly move the vertices of the triangulation so that no singular point is a vertex or lies on an edge. Then each singular point will lie inside a 2-simplex containing no other singular points.

8.3. The Poincaré Index Theorem

Henri Poincaré proved the following beautiful theorem, establishing a deep connection between the character of singular points of vector fields and the topology (as expressed by the Euler characteristic) of the surface on which they are defined.

Theorem 8.1. *The index of any smooth generic vector field on a (closed compact connected triangulated) orientable surface is equal to the Euler characteristic of this surface.*

Proof. The proof will be in two parts. In the first part, we will construct a special vector field whose index is indeed equal to the Euler characteristic of the surface. In the second part, we will prove that all generic vector fields on a given surface have the same index.

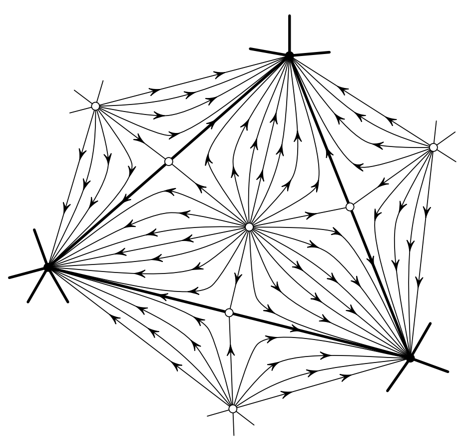

FIGURE 8.1. Singular points of the special vector field

Part 1. Let us fix a triangulation of our surface M. We will construct a special continuous vector field on the triangulated surface with singular points at all the vertices, at the midpoints of all the edges, and at the barycenters of all the faces, such that the index of this vector field is equal to the Euler characteristic of M. At the midpoint of each edge, we place a saddle point whose ingoing separatrix goes along the edge and whose outgoing separatrix goes to the barycenters of the two triangular faces adjacent to the edge. At each vertex, we place an unstable node so that the edges issuing from the vertex are covered by outgoing trajectories of the node. At the barycenter of each face, we place a stable node so that its ingoing trajectories include the three separatrices coming to the barycenter from the three saddle points at the midpoints of the face's three sides (Fig. 8.1). Finally, it is easy to see that the vector fields already constructed in the neighborhoods of the three types of points (vertices, midpoints, barycenters) can be extended continuously so as to cover the entire surface.

The index of the vector field thus constructed is obviously equal to the Euler characteristic $\chi = V - E + F$ of the surface. Indeed, the nodes at the vertices and the baricenters have index equal to $+1$, so that the nodes contribute $V + F$ to the index, while the saddle points have index equal to -1, so they contribute $-E$, and all that adds up to χ.

Part 2. Let V_1 and V_2 be two generic vector fields on our surface; our aim is to prove that they have the same index. First, by using Lemma 8.2, we can assume that all the singular points of V_1 and V_2 lie inside the 2-simplices (triangles) of the triangulation, no more than one in each. Next, by applying Lemma 8.1 at each vertex, we can assume that the vectors $V_1(a)$ and $V_2(a)$ have the same direction at each vertex a.

Now let us fix an orientation of M. Then each edge ab acquires two opposite orientations, ab and ba, from the two faces adjacent to it. Let a mobile point x move from a to b and then back to a; as x moves from a to b, consider the rotation of the vector issuing from a and equal to $V_1(x)$ followed by the rotation of the vector issuing from a and equal to $V_2(x)$ as the point x moves back from b to a; denote by d_{ab} the number of revolutions performed by the vector (d_{ab} is a well defined integer, because the two vector fields coincide at the vertices). In a similar way, we can define d_{ba}. Obviously, $d_{ab} = -d_{ba}$. Summing over the set E of all edges, we obtain

. $$\sum_{(ab)\in E} (d_{ab} + d_{ba}) = 0. \qquad (*)$$

Next let us look at this sum from the point of view of the set F of faces. Let $(abc) \in F$, where the cyclic order a, b, c agrees with the chosen orientation of M. Now consider the sum $d_{ab} + d_{bc} + d_{ca}$; it does not change if we first perform the rotation of all the vectors V_1 and then of all the vectors V_2; therefore,

$$d_{ab} + d_{bc} + d_{ca} = \mathrm{Ind}(\langle abc\rangle, V_1) + \mathrm{Ind}(\langle bac\rangle, V_2) = \mathrm{Ind}(\langle abc\rangle, V_1) - \mathrm{Ind}(\langle abc\rangle, V_2), \qquad (**)$$

where $\langle abc\rangle$ denotes the (positively oriented) closed curve bounding the face (abc). Rewriting the sum (*) as a sum over the faces, using (**), and Theorems 7.1 and 7.2, we obtain:

$$
\begin{aligned}
0 &= \sum_{(abc)\in F} \big(\mathrm{Ind}(\langle abc\rangle, V_1) - \mathrm{Ind}(\langle abc\rangle, V_2)\big) \\
&= \sum_{(abc)\in F} \mathrm{Ind}(\langle abc\rangle, V_1) - \sum_{(abc)\in F} \mathrm{Ind}(\langle abc\rangle, V_2) \\
&= \mathrm{Ind}(M, V_1) - \mathrm{Ind}(M, V_2). \qquad (**)
\end{aligned}
$$

The theorem is proved.

8.4. Applications

Here we state only two immediate applications of Poincaré's Theorem (there will be more in the exercise classes).

Corollary 8.1. *Any generic smooth vector field on the sphere has at least two singular points.*

Corollary 8.2. *Any smooth force field on the configuration space of the pentagonal linkage with fixed hinges at the distance* 3.9 *from each other and* 4 *mobile sides of length* 1 *has at least two equilibrium points.*

8.5. Exercises

8.1. On the torus construct a vector field without singular points.

8.2. On the Klein bottle construct a vector field without singular points.

8.3. On the sphere construct a vector field with one generic singular point.

8.4. On the projective plane construct a vector field with one singular point.

8.5. On the projective plane, does there exist a vector field (a) without any singular points, (b) with two generic singular points, (c) with three generic singular points, (d) with 17 generic singular points?

8.6. On the sphere with two handles construct a vector field with one singular point.

8.7. To each point X on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ a nonzero vector $v(X)$ in space is assigned. The vector depends continuously on the point of the sphere, but is not necessarily tangent to it. Prove that at least one of the vectors $v(X)$ is perpendicular to the tangent plane to the sphere at the point X.

8.8. Let $f\colon \mathbb{S}^2 \to \mathbb{S}^2$ be a continuous map. Prove that there exists a point $x \in \mathbb{S}^2$ such that $f(x) = \pm x$.