SU AND Sp BORDISM RINGS.

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The suggested project is devoted to a classical problem of description SU- and Sp-bordism rings.

Bordism and cobordism theory was actively developed in the 1950–1960s. Most leading topologists of the time have contributed to this area. The idea of bordism was first explicitly formulated by Pontryagin [Po] who related the theory of framed bordism to stable homotopy groups of spheres using the concept of transversality. Key results of bordism theory were obtained in the works of Rokhlin [Ro], Thom [Th], Novikov [No1, No2], Wall [Wa1], Averbuch [Av], Milnor [Mi2], Atiyah [At].

Topologists have quickly realised the potential of the Adams spectral sequence [Ad1] for calculations in bordism theory. It culminated with the description of the complex (or unitary) bordism ring $\varOmega^U$ in the classical works by Milnor [Mi2] and Novikov [No1, No2]. The ring $\varOmega^U$ was shown to be isomorphic to a graded integral polynomial ring $\mathbb{Z}[a_i\colon i \geqslant 1]$ with infinitely many generators, more precisely, with one generator in every even degree, $\deg a_i = 2i$. This result has since found numerous applications in algebraic topology and beyond.

In Novikov's 1967 work [No3] a brand new approach to cobordism and stable homotopy theory was proposed, based on the application of the Adams–Novikov spectral sequence and formal group laws techniques. This approach was further developed in the context of bordism of manifolds with singularities in the works of Mironov [Mir], Botvinnik [Bo] and Vershinin [Ve]. The Adams–Novikov spectral sequence has also become the main computational tool for stable homotopy groups of spheres [Ra].

As an illustration of his approach, Novikov outlined a complete description of the additive torsion and the multiplicative structure of the SU-bordism ring $\varOmega^{SU}$, which provided a systematic view on earlier geometric calculations with this ring. A modernised exposition of this description is given in the survey paper [CLP] by Chernykh, Limonchenko and Panov.

The structure of the Sp-bordism ring is a complete mystery. Using the classical Adams spectral sequence Novikov [No2] computed $\varOmega^{Sp}$ with 2 inverted:

$$\varOmega^{Sp} \otimes \mathbb{Z}\big[\tfrac{1}{2}\big] \cong \mathbb{Z}\big[\tfrac{1}{2}\big]\,[z_1, z_2, \ldots], \quad \deg z_i = 4i.$$

The Sp-bordism ring contains elements of any 2-power order [BK]. There are indecomposable Ray elements $\varphi_i \in \varOmega^{Sp}_{8i-3}$ of order 2 (cite [Ray]). But, unfortunately, more or less nothing is known about higher 2-torsion.

The general scope of the project consists in advancing the computation tools for the SU- and Sp-bordism rings.

***SU*-bordism.** I plan to present a "purely" algebraic computation of the SU-bordism ring using the homological version of the Adams-Novikov spectral sequence. The first step would consist in computing $MU_*(MSU)$ as a $MU_*(MU)$-comodule. The $MU_*(MSU)$ are explicitly present in the second sheet of the Adams-Novikov spectral sequence

$$E_2^{s,t} \cong \mathrm{Cotor}^s_{MU_*(MU)}(MU_*(pt), MU_*(MSU))_t$$

and a complete computation of E_2 would reduce the initial problem to a hard technical algebraic computation inside the spectral sequence.

***Sp*-bordism.** It seems worthwhile to investigate the application of the $MU_*(MU)$-Adams-Novikov spectral sequence to the Thom spectrum MSp as it may give new information about the Sp-bordism ring. The first step is a computation of $MU_*(MSp)$ as $MU_*(MU)$-comodule. As in the last problem, this can be applied to make algebraic computations in the corresponding Adams-Novikov spectral sequence.

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