# Summary of the research project: "On different non-Abelian analogs of the differential Painlevé equations"

Irina Bobrova

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The celebrated six differential Painlevé equations have appeared as a result of a classification of complex second-order differential equations of the form

$$y''(z) = P\left(z, y(z), y'(z)\right), \qquad y(z),\ z \in \mathbb{C},$$

where $P\left(z, y(z), y'(z)\right)$ is a meromorphic function in $z$ and is a rational function in $y(z)$, $y'(z)$, that satisfy *the Painlevé property*. Their solutions are currently known as the most general class of special functions called *the Painlevé transcendents*. They arise in a wide range of applications in mathematics and physics and have surprisingly rich mathematical structures.

Quantum, or more generally, non-commutative extensions of various integrable systems have acquired considerable attention in recent years. This interest was motivated by problematics and needs of modern quantum physics as well as by a natural attempts of mathematicians to extend and to generalise the "classical" integrable structures and systems. The Painlevé transcendents provide a good example of this phenomena. For instance, some quantum analogs were appeared in [Nagoya et al., 2008]. An interesting non-commutative family of non-autonomous many-particle integrable systems was introduced in [Bertola et al., 2018], who defined an isomonodromic representation of Takasaki Hamiltonian systems of Calogero-Painlevé families [Takasaki, 2001].

Our aim is to discover integrable non-abelian and matrix analogs of the scalar differential Painlevé equations, using two different approaches. One of them has been developed by Vladimir Sokolov [Balandin and Sokolov, 1998] and another one by Vladimir Retakh and Vladimir Rubtsov [Retakh and Rubtsov, 2010].

In **the first approach**, the dependent variable $y(z)$ belongs to the matrix algebra $\mathrm{Mat}_n(\mathbb{C})$ and the independent variable $z$ is an element of $\mathbb{C}$. By the matrix generalization of the Painlevé-Kovalevskaya test introduced in [Balandin and Sokolov, 1998], we are able to verify that a matrix ODE satisfy the Painlevé property (that is a necessary condition). By this method, it has been established several integrable matrix analogs for the first, second, and fourth Painlevé equations in the papers [Balandin and Sokolov, 1998], [Adler and Sokolov, 2021], [Bobrova and Sokolov, 2021], respectively. For example, the Painlevé-4 equation has three linearly non-equivalent analogs that have isomonodromic Lax representation (this part of the work is submitted to the arXiv). We are going to apply the Painlevé-Kovalevskaya test for searching new examples of matrix generalizations for other Painlevé equations and construct isomonodromic Lax pairs. We also plan to describe for one or for several analogs their matrix hierarchies, Hamiltonian structures, symmetries, and non-commutative monodromy surfaces.

In **the second approach** developed by Vladimir Retakh and Vladimir Rubtsov, the variables $y(\bar{z})$, $\bar{z}$ are elements of the unital associative algebra $\mathcal{A}_{\mathbb{C}}$ with division and a derivation such that $\bar{z}' = 1$ and $\alpha' = 0$ for any $\alpha \in \mathbb{C}$. In their paper [Retakh and Rubtsov, 2010], the authors have constructed the so-called fully non-commutative analog for the $\mathrm{P}_2$ equation and discovered its solutions in terms of the quasideterminants in the almost Hankel form. The fully non-commutative $\mathrm{P}_2$ equation is also integrable as it has an isomonodromic representation found in [Irfan, 2012]. By this approach we are going to derive the fully non-commutative non-stationary $\mathrm{P}_2^{(n)}$ hierarchy, describe its symmetries and solutions to generalize results from the paper [Bobrova, 2020] to the non-commutative case. We also plan to describe a structure of the solutions of the fully non-commutative analog of the Painlevé-4 equation $\mathrm{P}_4^0$, found in [Bobrova and Sokolov, 2021], namely their quasideterminant representation in the almost Hankel form. This point is a generalization of the results from the paper [Joshi et al., 2006] to the non-commutative case.

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