Report on "Young Russian Mathematics" award

Irina Bobrova

2023 year

Contents

1 Results

1.1 A full classification of non-Abelian Painlevé systems with abelian parameters

All Painlevé equations are Hamiltonian and can be written as a coupled system of first-order ordinary differential equations [Malmquist, 1922], [Okamoto, 1980]. These systems are non-autonomous and, therefore, the Hamiltonians are not conserved quantities. Since well-known properties of integrable autonomous systems are trivially generalized to the non-commutative case [Mikhailov and Sokolov, 2000], it is convenient to consider auxiliary autonomous systems related to the Painlevé systems. Recently, using this approach, we have constructed two classes (Hamiltonian systems [Bobrova and Sokolov, 2023a] and systems with Okamoto integral [Bobrova and Sokolov, 2023b]) of non-abelian Painlevé type systems that are closed under the limiting transitions and some group-actions. These two criteria are useful to discover some known and several new examples of non-abelian Painlevé systems. However, they do not detect **all** known examples of non-abelian P_4 and P_2 systems (see [Bobrova and Sokolov, 2022] and [Adler and Sokolov, 2021]) found by the matrix generalization of the Painlevé-Kovalevskaya test [Balandin and Sokolov, 1998]. Due to this fact, we turn to the search for such a criterion that will select, in particular, all known non-abelian Painlevé systems with abelian parameters. It turns out that the symmetry approach is useful in this classification problem which was done in the paper [Bobrova and Sokolov, 2023c].

We have found a complete list of the Painlevé-6 type non-abelian systems whose auxiliary autonomous systems possess a commuting symmetry. Due to the complexity of the calculations, we first classify homogeneous systems, which are the leading parts of autonomous systems of the P_6 type. For each leading part, finding the remaining coefficients is not too difficult. The resulting list contains 35 different systems, including 19 systems from [Bobrova and Sokolov, 2023a], [Bobrova and Sokolov, 2023b]. One of the reasons for the appearance of a large number of non-abelian systems of the same type is the presence of transformations that preserve the integrability. Proceeding to other cases, we have also found all non-abelian $P_5 - P_1$ systems admitting auxiliary autonomous systems with a symmetry.

The obtained set of systems is closed under the degeneracy and some group-actions and has an isomonodromic representation. As a result, the considered criterion allows us to find all integrable non-abelian Painlevé systems with abelian parameters.

1.2 Non-Abelian monodromy surfaces

The Painlevé equations are connected with a system of scalar differential equations [Fuchs, 1907], [Garnier, 1912] integrable in the sense of the Frobenius theorem. In the paper [Jimbo and Miwa, 1981b], it was established that the Painlevé equations can be linearized. This fact is connected with monodromy preserving deformations related to vector bundles of rank 2. Due to the isomonodromic property, the space of solutions of the Painlevé equations can be parameterized by the monodromy data. Namely, each of the equations can be associated with the zero-locus of an affine cubic that is usually called *the monodromy surface* (e.g., [Van Der Put and Saito, 2009]). We are interested in non-abelian generalizations of the well-known monodromy surfaces related to different linearizations of the non-commutative analogs for the second Painlevé equation, obtained in [Adler and Sokolov, 2021] and labeled by P_2^0, P_2^1, P_2^2.

In the commutative case, the P_2 equation has two monodromy surfaces of the FN-type and JM-types. The corresponding linearizations are related to each other by a generalized Laplace transformation [Suleimanov, 2008], [Joshi et al., 2009]. We have extended this correspondence to the non-commutative case and, as a result, presented for each of the non-abelian P_2 systems linearizations of the FN and JM types.

To proceed to the non-abelian monodromy surfaces, we need to determine the monodromy data consisting of the topological monodromy, the formal monodromy, and the Stokes matrices. This data is defined by a formal solution near a singular point. Due to this, we have studied such a formal solution and have derived a statement, that is a non-abelian generalization of Proposition 2.2 in [Jimbo and Miwa, 1981a] and a generalization of Proposition 4.1 in [Bertola et al., 2018].

Regarding the case of the P_2^0 system, the monodromy data is isomonodromic and, thus, the corresponding monodromy surfaces were derived [Bobrova, 2023]. In the commutative setting, they coincide with the well-known. Note also that in the commutative case these surfaces are equivalent by a simple scaling that cannot be generalized to the non-abelian setting. Regarding the remaining systems, P_2^1 and P_2^2, the monodromy data are not isomonodromic and, therefore, we cannot parameterize their solutions by the Stokes multipliers. But, in fact, one can ask about a gauge-transformation that makes the monodromy data isomonodromic. As far as we know, such a transformation does not exist.

1.3 Some non-Abelian discrete Painlevé equations

We are interested in an extension to the non-commutative case of the following scheme

$$\begin{array}{ccc} \text{discrete Toda equations} & \xrightarrow{\text{[Hone et al., 2017]}} & \text{Somos-}N\text{ equations} \\ & \xrightarrow{\text{[Hone and Inoue, 2014]}} & \text{discrete Painlevé equations.} \end{array}$$

For this purpose, by using a non-abelian Jacobi identity for the quasideterminants [Gelfand and Retakh, 1991], we have derived a non-commutative two-dimensional discrete Toda lattice, the 2ddTL [Bobrova et al., 2023], differ from [Nimmo, 2006]. Our analog possesses Lax pairs of different types and solutions in terms of quasideterminants. Note that its form is more suitable for the further reductions to (1+1)-dimensional and one-dimensional systems. In particular, plane-wave reductions of the 2ddTL and its scalar Lax pair lead to a non-commutative analog for the autonomous Somos-N like sequences and the corresponding isospectral pairs. Note that commutative Somos-N like sequences are associated with $q\mathrm{P}_1$ and $q\mathrm{P}_2$ equations and their hierarchies [Hone and Inoue, 2014]. In order to generalize this connection to the non-abelian case, one needs to get a non-autonomous analog for Somos-N like sequences. We suggest a method which in the commutative case is different from those in [Hone and Inoue, 2014]. This allows us to obtain a non-abelian non-autonomous Somos-N like equation and, as a result, non-commutative analogs for the q-Painlevé-1 and q-Painlevé-2 equations and their hierarchies. Note that the problem of finding the corresponding Lax pairs for these non-autonomous systems remains open even in the commutative case.

2 Papers

2.1 Published

1. Bobrova I., Sokolov V. *On classification of non-abelian Painlevé type systems*, Journal of Geometry and Physics, 191: 104885, arXiv:2303.10347

2. Bobrova I. *Different linearizations of non-abelian second Painlevé systems and related monodromy surfaces*, Journal of Mathematical Physics, 64(10): 101702, arXiv:2302.10694

3. Bobrova I., Sokolov V. *Classification of Hamiltonian non-abelian Painlevé type systems*, Journal of Nonlinear Mathematical Physics, 30: 646–662, arXiv:2209.00258

4. Bobrova I., Sokolov V. *Non-abelian Painlevé systems with generalized Okamoto integral*, in *The Diverse World of PDEs: Algebraic and Cohomological Aspects*, volume 789 of *Contemporary Mathematics*, pages 41–76, American Mathematical Society, arXiv:2206.10580

2.2 Submitted

1. Bobrova I., Retakh V., Rubtsov V., Sharygin G. *Non-Abelian discrete Toda chains and related lattices*, Physica D: Nonlinear Phenomena (under review), arXiv:2311.11124

3 Scientific conferences and schools

3.1 Scientific conferences

1. Forum des jeunes mathématicien-nes. Analyse. Géométrie. Application, Bruxelles, November 22 - 24
talk *Non-Abelian Painlevé equations and related monodromy surfaces*

3.2 Schools

1. *Atelier on Higher Structures in Differential Geometry*, Lyon, Institut Camille Jordan

4 Work in scientific centers and international groups

1. Invited Researcher, IHES (2 dec 2022 – 28 feb 2023)

2. Junior Research Fellow, International Laboratory of Cluster Geometry, Faculty of Mathematics, HSE University (10 mar 2023 – 31 oct 2023)

5 Teaching

5.1 Courses 2022/2023 (spring)

1. Seminars on Fourier Analysis, 2d year of Bachelor's programme "Mathematica", Faculty of Mathematics, HSE University (only in January)

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[Bobrova and Sokolov, 2023b] Bobrova, I. and Sokolov, V. (2023b). Non-abelian Painlevé systems with generalized Okamoto integral. In *The Diverse World of PDEs: Algebraic and Cohomological Aspects*, volume 789 of *Contemporary Mathemaics*, pages 41–76. American Mathematical Society. arXiv:2206.10580.

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