Introduction into (projective) algebraic geometry

Course program, variant 1, 2020

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The base field k is algebraically closed, unless specified otherwise. Characteristic is any, but sometimes we will have to assume that it is large enough (usually 5 is enough), or zero.

- Introduction
  - Affine and projective varieties - definition and a few examples. Projective spaces. Veronese embedding.
- Algebraic curves - 1.
  - Plane curves, smooth and singular.
  - Plane conics are rational.
  - Smooth plane cubics are not rational.

    Singular plane cubics.

    The group law on a plane cubic (elliptic curve). Case of the field of complex numbers (statement only).
  - Genus of a curve, definition via Poincare polynomial.
  - Arithmetic genus of a smooth curve via cohomology, H^1(O) [1] . Equivalence with the POincare series definition.
  - Linear equivalence of divisors on a curve. Picard group. Picard group of P^1. Elliptic curve: detailed study of linear systems.
  - Differential forms on curves.
  - Residues on a curve: statement only.
  - Serre duality: sketch of the proof with resudes [2].
  - Riemann-Roch theorem: sheaf-theoretic proof [3] .
  - Picard variety: existence: statement only. Morphism from a symmetric power of a curve to the Picard variety.

[1] If I can assume that the notion of sheaf cohomology is known. I only need H^1 of a sheaf. Cech definition is enough.

[2] same

[3] same

- Hurwitz formula: algebraic proof.
- Case of complex numbers: Hurwitz formula: topological proof.
- Canonical embedding of a curve. Example: curves of genus 3 are plane curves of degree 4.
- Riemann-Roch theorem in a geometric form: linear span of a set of points in the canonical embedding.
- Hyper-elliptic curves. Curves of genus 2 are hyper-elliptic.
- Zeta-function of a curve over a finite field: defintion, and statement of the Weil's conjecture.

- Algebraic surfaces - 1.
  - Blow-up of a point on a plane.
  - Adjunction formula for curves on a surface. Genus of a plane curve, again.
  - Intersection index of properly intersecting curves.
  - Picard group of a surface: definition. Picard group of P^2, and P^1 x P^1.
  - First Chern class of a line bundle on sa surface (with values in the Picard group.)
  - Difficult part: self-intersection of a curve. Examples: exceptional curve of the blow-up; resolution of a cone singularity.
  - Neron-Severi group of a surface. Examples: P^2, quadric.
  - Digression: resolution of singularities of surfaces – some examples.
  - Cubic surfaces in P^3.
    * Linear system of plane cubic curves through 6 points on P^2, and 27 lines: sketch of a proof. Picard group of a cubic surface is isomorphic to E_6.
  - Cremona transformations of P^2: Example.
- Grassmannian varieties - 1.
  - Grassmanian variety Gr(2,4).
  - Projective embedding of Gr(2,4). Plucker quadric.
  - Intersection theory on Gr(2,4). 27 lines, again. Idea of rational equavlence of cycles.